Proceedings of Symposia in Pure Mathematics

Lectures on Lyapunov Exponents and Smooth Ergodic Theory

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Contents

2000 *Mathematics Subject Classification.* Primary: 37D25, 37C40.

Key words and phrases. Lyapunov exponents, nonuniformly hyperbolic dynamical systems, smooth ergodic theory.

L. Barreira was partially supported by FCT's Funding Program and the NATO grant CRG 970161. Ya. Pesin was partially supported by the National Science Foundation grant #DMS-9704564 and the NATO grant CRG 970161.

Introduction

This manuscript is based on lectures given by Ya. Pesin at the AMS Summer Research Institute (Seattle, Washington, 1999). It presents the core of the nonuniform hyperbolicity theory of smooth dynamical systems. This theory was originated in [**26, 27, 28, 29**] and has since become a mathematical foundation for the paradigm which is widely-known as "deterministic chaos" — the appearance of irregular "chaotic" motions in pure deterministic dynamical systems. We follow the original approach by Ya. Pesin making some improvements and necessary modifications.

The nonuniform hyperbolicity theory is based on the theory of Lyapunov exponents which was originated in the works of Lyapunov [**19**] and Perron [**25**] and was developed further in [**7**]. We provide an extended excursion into this theory. This includes the abstract theory of Lyapunov exponents — that allows one to introduce and study the crucial concept of Lyapunov-Perron regularity (see Section 2) — as well as the advanced stability theory of differential equations (see Sections 1, 3, and 4).

Using the language of the theory of Lyapunov exponents one can view nonuniformly hyperbolic dynamical systems as those where the set of points whose Lyapunov exponents are *all* nonzero is "large", for example, has full measure with respect to an invariant Borel measure (see Sections 6 and 7). In this case the fundamental Multiplicative Ergodic theorem of Oseledets [**24**] implies that almost every point is Lyapunov–Perron regular. Thus, the powerful theory of Lyapunov exponents applies and allows one to carry out a thorough analysis of the local stability of trajectories.

The crucial difference between the classical uniform hyperbolicity and its weakened version of nonuniform hyperbolicity is that the hyperbolicity conditions can get worse when one moves along the trajectory of a nonuniformly hyperbolic point. However, if this point is Lyapunov–Perron regular then the worsening occurs with subexponential rate and the contraction and expansion along stable and unstable directions prevail.

One of the crucial manifestations of this fact is the fundamental Stable Manifold theorem that was established in [**27**] and is a generalization of the classical Hadamard–Perron theorem. In Section 9 we present the proof of the Stable Manifold theorem following the original approach in [**27**] which is essentially an elaboration of the Perron method. In Section 10 we sketch the proof of a slightly more general version of the Stable Manifold theorem (known as the Graph Transform Property) which is due to Hadamard. We also describe several main properties of local stable manifolds of which one of the most important is that their sizes may decrease along trajectories only with subexponential rate (and thus the contraction prevails).

There are several methods for establishing nonuniform hyperbolicity. One of them, which we consider in these lectures, is to show that the Lyapunov exponents of the system are nonzero. One of the first examples was constructed in [**26**]. It is a three dimensional flow on a compact smooth Riemannian manifold which is a reconstruction of an Anosov flow. It reveals some mechanisms for the appearance of zero Lyapunov exponents (see the discussion in Section 8).

Another example are geodesic flows on compact smooth Riemannian manifolds of nonpositive curvature. Let us stress that geodesic flows have always been a good source of examples and have provided inspiration for developing the hyperbolicity theory. In these lectures we consider only geodesic flows on surfaces of nonpositive curvature and show that they are nonuniformly hyperbolic on an open and dense set (see Section 8).

One of the main goals of the nonuniform hyperbolicity theory is to describe the ergodic properties of a smooth dynamical system preserving a smooth invariant measure. This is done in Sections 11, 12, 13, and 15 where we show that such a system has ergodic components of positive measure and also establish the entropy formula that expresses the entropy of the system via its positive Lyapunov exponents. Finally, in Section 16 we apply these results to the geodesic flows on compact surfaces of nonpositive curvature.

Acknowledgment. We want to express our deep gratitude to M. Brin, D. Dolgopyat, C. Pugh, and J. Schmeling for their valuable comments and discussions.

The lectures were used as a handout for a graduate course on Dynamical Systems with Nonzero Lyapunov Exponents that Ya. Pesin taught during the Fall 2000 semester at The Pennsylvania State University. Ya. Pesin would like to thank students for their patience during the course and numerous fruitful remarks which helped improve the text. The authors are specially grateful to students C. Carter, T. Fisher, R. Gunesch, I. Ugarcovici, and A. Windsor for many valuable comments and corrections.