THE ESSENTIAL COEXISTENCE PHENOMENON IN DYNAMICS

JIANYU CHEN, HUYI HU, AND YAKOV PESIN

Abstract. This article is a survey of recent results on the essential coexistence of hyperbolic and non-hyperbolic behavior in dynamics. Though in the absence of a general theory, the coexistence phenomenon has been shown in various systems during the last three decades. We will describe the contemporary state of the art in this area with emphasis on some new examples in smooth conservative systems, in both cases of discrete and continuous-time.

1. Introduction

The problem of essential coexistence of regular and chaotic behavior lies in the core of the theory of smooth dynamical systems. We stress that through out this paper we consider dynamical systems on compact smooth Riemannian manifolds with discrete and continuous time – diffeomorphisms and flows – that preserve a smooth measure, i.e., a measure that is equivalent to the Riemannian volume m.

The early stages in the development of the theory of dynamical systems were dominated by the study of regular dynamics including presence and stability of periodic motions, translations on surfaces, etc.

Already Poincaré in 1889 discovered existence of homoclinic tangles in conjunction with his work on the three-body problem. However, an intensive rigorous study of chaotic behavior in purely deterministic smooth dynamical systems began in the second part of the last century due to the pioneering work of Anosov, Sinai, Smale and others. This has led to the development of hyperbolicity theory in its three main incornations: uniform, nonuniform and partial hyperbolicity.

It was therefore natural to ask whether the two types of dynamical behavior – regular and chaotic – can coexist in an essential way. Let us stress that coexistence phenomenon can already be observed in systems that are non-uniformly hyperbolic: while the Lyapunov exponents along the majority of trajectories are all nonzero, there must exist trajectories along which some or all Lyapunov exponents are zero. However, the latter forms a set of zero volume and hence can be "neglected".

There are different ways in how the coexistence phenomenon can be stated and in this paper we adapt the following approach. Recall that the Lyapunov exponent of f at a point $x \in \mathcal{M}$ of a vector $v \in T_x\mathcal{M}$ is defined by the formula

$$\lambda(x, v, f) = \limsup_{n\to\infty} \frac{1}{n} \log \|d_x f^n v\|.$$

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Presence of zero Lyapunov exponents on a subset of positive volume indicates a certain regular behavior of f, while presence of nonzero Lyapunov exponents on a subset of positive volume is a sign of certain level of chaotic behavior of the system.

1.1. **The coexistence phenomenon.**

Definition 1.1. *We say that a diffeomorphism f of a compact smooth manifold $\mathcal{M}$ exhibits an essential coexistence of regular and chaotic behavior if*

(1) the manifold $\mathcal{M}$ can be split into two f-invariant disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of positive volume – the chaotic and regular regions for f;[1]
(2) for almost every $x \in \mathcal{A}$ the Lyapunov exponents at x are all nonzero;
(3) $f|\mathcal{A}$ is ergodic;[2]
(4) for every $x \in \mathcal{B}$ the Lyapunov exponents at x are all zero;
(5) the set $\mathcal{A}$ is dense in $\mathcal{M}$.

The last requirement means that the regular and chaotic regions for f cannot be topologically separated. In this case we say that we deal with essential coexistence of type I. If Condition 5 is dropped we say that we have essential coexistence of type II. We shall see examples of essential coexistence of both types below.

There is a weaker version of the of the above definition in which one replaces Condition 2 with the requirement that some (but may be not all) of the Lyapunov exponents are nonzero (see the article [37] where some results on essential coexistence in this case are surveyed).

Our definition of the essential coexistence phenomenon is inspired by a discrete version of the classical KAM phenomenon in the volume preserving category as described in the work of Cheng and Sun [10] (in the three dimensional case) and of Herman [24], Xia [40] and Yoccoz [41] (in the general case). We follow the approach in [40].

Consider a family of volume preserving diffomorphisms $F_\varepsilon : (r, \theta_1, \dots, \theta_n) \mapsto (r', \theta_1', \dots, \theta_n')$ from $[1, 2] \times \mathbb{T}^n$ to $\mathbb{R} \times \mathbb{T}^n$ where

$$\begin{aligned} r' &= r + \varepsilon f_0(r, \theta_1, \dots, \theta_n), \\ \theta_1' &= \theta_1 + g_1(r) + \varepsilon f_1(r, \theta_1, \dots, \theta_n), \\ &\vdots \\ \theta_n' &= \theta_n + g_n(r) + \varepsilon f_n(r, \theta_1, \dots, \theta_n), \end{aligned}$$

$g_1, \dots, g_n$ and $f_0, f_1, \dots, f_n$ are real analytic functions in all of their variables and ε is a small perturbation parameter. Note that the map F_0 preserves every torus $\{r\} \times \mathbb{T}^n$, $r \in [1, 2]$.

Theorem 1.2. *Assume that the functions $g_1, \dots, g_n$ satisfy the* twist condition*, that is for all $r \in [1, 2]$*

$$\det(g_i^{(j)}(r)) \geq d > 0,$$

where $g_i^{(j)}(r)$ is the j-th derivative of $g_i(r)$, $1 \leq i, j \leq n$, and $d > 0$ is a constant. Then for any sufficiently small $\varepsilon > 0$, there are a Cantor set $S(\varepsilon) \subset [1, 2]$ and a map $\psi : S(\varepsilon) \to C^\omega(\mathbb{T}^n, \mathbb{R} \times \mathbb{T}^n)$ such that the graph of $\psi(r)$, for each $r \in S(\varepsilon)$, is an invariant torus under the map F_ε. Furthermore, the map F_ε induced on each torus can be parameterized as a Diophantine translation. Finally, the Lebesgue measure of the set $S(\varepsilon)$ tends to 1 as $\varepsilon \to 0$.

[1] In many interesting examples the set $\mathcal{A}$ is open, see below.

[2] Note that since the Lyapunov exponents at almost every $x \in \mathcal{A}$ are all nonzero, the map $f|\mathcal{A}$ has at most countably many ergodic components of positive measure (see [4, 30]).

The above result still holds if the functions $f_0, f_1, \ldots, f_n$ are assumed to be of class C^τ for any $\tau > 2n+1$. As observed in [24,41], one can embed the product space $[1,2] \times \mathbb{T}^n$ into an arbitrary compact smooth manifold $\mathcal{M}$ of dimension $n+1$ and extend the unperturbed map F_0 to a diffeomorphism $\widetilde{F}_0$ of $\mathcal{M}$. Thus there exists an open neighborhood U of $\widetilde{F}_0$ in the C^τ topology such that any volume preserving diffeomorphism $P \in U$ possesses a set of codimension-1 invariant tori of positive volume. Moreover, on each such torus the diffeomorphism P is C^1 conjugate to a Diophantine translation. Furthermore, all Lyapunov exponents of P are zero on the invariant tori.

Since the set of invariant tori is nowhere dense, it is expected that it is surrounded by "chaotic sea", that is outside this set the Lyapunov exponents are all nonzero and the system is ergodic. It has since been an open problem to find out to what extend this picture is true.

1.1.1. *Essential coexistence of type I.* A first step towards understanding this picture is to construct a particular example of a volume preserving diffeomorphism exhibiting essential coexistence phenomenon in the spirit of Definition 1.1.

Theorem 1.3 (Hu, Pesin and Talitskaya, [27]). *Given $\alpha > 0$, there exists a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 5 and a C^∞ volume preserving diffeomorphism $P : \mathcal{M} \to \mathcal{M}$ such that*

(1) $\|P - Id\,\|_{C^1} \leq \alpha$;
(2) P is ergodic on an open and dense subset $\mathcal{U} \subset \mathcal{M}$ with $m(\mathcal{U}) < m(\mathcal{M})$; in particular, P is topologically transtive on $\mathcal{M}$; furthermore, $P|\mathcal{U}$ is a Bernoulli diffeomorphism;
(3) the Lyapunov exponents of P are nonzero for almost every $x \in \mathcal{U}$;
(4) the complement $\mathcal{U}^c$ has positive volume, $P|\mathcal{U}^c = Id$ and the Lyapunov exponents of P on $\mathcal{U}^c$ are all zero.

The regular region $\mathcal{U}^c$ is a Cantor set of invariant submanifolds. More precisely, $\mathcal{U}^c$ is the direct product $\mathcal{N} \times C$, where $\mathcal{N}$ is a 3-dimensional smooth compact manifold and C is a Cantor set of positive Lebesgue measure in the 2-torus $\mathbb{T}^2$. Thus $\mathcal{U}^c$ has codimension two. We shall explain the construction of this theorem in Section 3.

By modifying the construction in Theorem 1.3, one can obtain a C^∞ diffeomorphism P of a compact smooth Riemannnian manifold of dimension 4 with similar properties (Chen, [8]). In this example, the regular region $\mathcal{U}^c$ is the direct product of a 3-dimensional compact manifold and a Cantor set of positive length in a circle and thus has codimension one. As a result, the map P has countably many ergodic components in the chaotic region $\mathcal{U}$.

1.1.2. *Essential coexistence of type II.* The fact that the set $\mathcal{U}$ is dense in $\mathcal{M}$ means that the situation described in Theorem 1.3 presents essential coexistence of type I and makes it significantly different from other examples demonstrating the coexistence phenomenon of type II in which the chaotic and regular regions are topologically separated. Examples of coexistence phenomenon of type II have been constructed and intensively studied in dimension 2.

Przytycki [33] studied a C^∞-family of diffeomorphisms of $\mathbb{T}^2$, which demonstrates a route from uniform hyperbolicity to non-uniform hyperbolicity and then to coexistence of regular and chaotic behavior. More precisely, consider a smooth one-parameter family of C^∞ area preserving diffeomorphisms $H_\varepsilon : \mathbb{T}^2 \to \mathbb{T}^2$, $-\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$ given by

$$H_\varepsilon(x,y) = (x+y, y + h_\varepsilon(x+y)) \tag{1.1}$$

where h_ε is such that

(a) h_ε is an odd function for every ε;
(b) $h_\varepsilon(0) = 0$, $h_\varepsilon(1) = 1$, $h_0'(0) = h_0''(0) = 0$, $h_0^{(3)}(0) > 0$;
(c) $\frac{d}{d\varepsilon} h_\varepsilon'(0) > 0$ and $h_0'(x) > 0$ for all $x \neq 0$.

One can show that

(1) for every $\varepsilon > 0$ the map H_ε is an Anosov diffeomorphism and for every $\varepsilon \geq 0$ the map H_ε is topologically conjugate to the hyperbolic automorphism given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$;
(2) the family H_ε at $\varepsilon = 0$ is transversal to the boundary $\partial\mathrm{An}$ of the set of Anosov diffeomorphisms of $\mathbb{T}^2$ in the sense that there is a constant $C > 0$ such that
$$\mathrm{dist}_{C^1}(H_\varepsilon, \partial\mathrm{An}) \geq C|\varepsilon|;$$
(3) for every $\varepsilon < 0$ there exists an elliptic island O_ε - a neighborhood around $0 \in \mathbb{T}^2$, most of which is filled with invariant closed curves.

Furthermore, for a specially chosen h_ε one can show that

(4) for every sufficiently small $\varepsilon < 0$ the elliptic island O_ε is the domain between the separatrices connecting two saddles near $0 \in \mathbb{T}^2$ and the map H_ε behaves stochastically on $S_\varepsilon := \mathbb{T}^2 \setminus \overline{O_\varepsilon}$, more exactly, the Lyapunov exponents for $H_\varepsilon|_{S_\varepsilon}$ are nonzero almost everywhere and $H_\varepsilon|_{S_\varepsilon}$ is isomorphic to a Bernoulli automorphism.

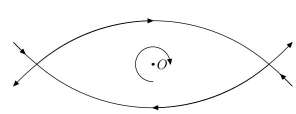

FIGURE 1. Separatrices

Note that the elliptic island O_ε and the chaotic sea S_ε are sharply separated by the separatrices, and therefore S_ε is not dense in $\mathbb{T}^2$. We point out that it is still unknown whether there exists a stochastic sea of positive measure for a general family H_ε (given by (1.1)) satisfying properties (a)–(c). By the entropy formula [4,30], this question is equivalent to the problem of whether H_ε has positive metric entropy. To this end Liverani [29] has shown that there exists a constant $C > 0$ and $\varepsilon_0 > 0$ such that for each $-\varepsilon_0 < \varepsilon < 0$ one can construct a C^∞ area preserving diffeomorphism $\widetilde{H}_\varepsilon : \mathbb{T}^2 \to \mathbb{T}^2$ with positive metric entropy such that
$$\mathrm{dist}_{C^2}(\widetilde{H}_\varepsilon, H_\varepsilon) \leq e^{-C\varepsilon^{-\frac{1}{2}}}, \quad m(\{x \in \mathbb{T}^2 : \widetilde{H}_\varepsilon(x) \neq H_\varepsilon(x)\}) \leq e^{-C\varepsilon^{-\frac{1}{2}}}.$$
This means that the new family $\widetilde{H}_\varepsilon$ approaches the family H_ε with an exponential rate.

1.2. **The standard map.** We consider the most famous example of symplectic twist maps - the Chirikov-Taylor standard map given by

$$f_\lambda(x,y) = (x+y, y+\lambda\sin 2\pi(x+y)), \quad \lambda\in\mathbb{R},$$

where the coordinates (x,y) run over either the cylinder or the 2-torus $\mathbb{T}^2$. This map was introduced and studied independently by Taylor (1968) and Chirikov (1969) using numerical methods available at the time. Their results were published about ten years later, see [11,21]. Since then the standard map has been studied intensively both numerically and theoretically but a complete description of its ergodic properties are nowhere in sight. It has been shown that for $\lambda\neq 0$ the map f_λ is non-integrable and possesses a horseshoe, see [1,20,22]. Hence, it has positive topological entropy.

Sinai [36] conjectured that the metric entropy $h_m(f_\lambda)$ with respect to the area m is positive for all $\lambda\neq 0$ and that $h_m(f_\lambda)$ grows to infinity as $\lambda\to\infty$. This conjecture is still open and turns out to be very difficult. One major obstacle is that the elliptic islands may be present for arbitrary large λ, and the set of elliptic islands seemingly forms a dense subset of large measure (see [11,16]), which makes it hard to find invariant cones to establish nonzero Lyapunov exponents. Nevertheless, various results in this direction for other but related surface diffeomorphisms have been obtained. One would often observe the coexistence of KAM invariant circles and the chaotic sea in these results.

1.2.1. *Piecewise linear perturbation of the standard map.* Note that by the symplectic coordinate change $q=x, p=x+y$, the standard map f_λ is equivalent to the map

$$\widetilde{f}_\lambda(q,p) = (p, -q+2p+\lambda\sin(2\pi p)).$$

In [38], Wojtkowski considered the map T_λ with $\sin(2\pi p)$ in the above formula replaced by the function

$$h(p) = \begin{cases} -p-\frac{1}{4}, & -\frac{1}{2}\le p\le 0 \\ p-\frac{1}{4}, & 0\le p\le \frac{1}{2} \end{cases}$$

One can view T_λ as a piecewise linear perturbation of the completely integrable map $\widetilde{f}_0$. For some specific values of λ, Wojtkowski showed that

(1) if $\lambda\ge 4$ then T_λ is almost hyperbolic in the 2-torus $\mathbb{T}^2$; moreover, T_λ is Bernoulli if $\lambda>\lambda_0\approx 4.0329$;
(2) if $\lambda = 2(\cos\frac{\pi}{n}+1)$, $n=2,3,\dots$, there is an elliptic island D around the elliptic fixed point $(-\frac{1}{4},\frac{1}{4})$ such that T_λ is almost hyperbolic in $\mathbb{T}^2\backslash D$. Furthermore, D is a $2n$-sided convex polygon centered at $(-\frac{1}{4},\frac{1}{4})$, and $T_\lambda|D$ has period n for n odd and period $2n$ for n even;
(3) for $\lambda=1$, the 2-torus is divided into two invariant parts M_1 and M_2. There is a hexagonal elliptic island D in M_1 such that T_1 is almost hyperbolic in $M_1\backslash D$.

Clearly, the last two cases demonstrate the coexistence of chaotic and integrable behavior. Wojtkowski [39] considered a sequence of arbitrarily small values of λ and showed that the stochastic sea exists and has asymptotic Lebesgue measure $\frac{1}{16}\lambda\log\frac{1}{\lambda}(1+o(1))$ as $\lambda\to 0$.

1.2.2. *Hausdorff dimension of the stochastic sea.* Duarte [16] showed that there exists $\lambda_0>0$ and a residual subset $\mathcal{R}\subset[\lambda_0,\infty)$ such that for every $\lambda\in\mathcal{R}$ there exists a basic set Ω_λ, i.e., a transitive locally maximal hyperbolic set, for the map f_λ satisfying:

(1) Ω_λ is dynamically increasing, i.e., $\Omega_{\lambda+\varepsilon}$ contains the continuation of Ω_λ at parameter $\lambda+\varepsilon$;

(2) Ω_λ is $\frac{4}{\lambda^{\frac{1}{3}}}$-dense;

(3) the Hausdorff dimension $\dim_H(\Omega_\lambda) \geq 2\dfrac{\log 2}{\log(2+\frac{9}{\lambda^{\frac{1}{3}}})}$;

(4) each point in Ω_λ is an accumulation point of elliptic islands of f_λ.

Gorodetski [23] found a transitive invariant set $\widetilde{\Omega}_\lambda$ containing Ω_λ such that the Hausdorff dimension $\dim_H(\widetilde{\Omega}_\lambda) = 2$. It is worth pointing out that the construction in [16,23] utilizes the unfolding of a homoclinic tangency, which can create similar scenarios for some other generic surface diffeomorphisms, such as the area preserving Hénon family [17,23].

1.3. **Other examples of coexistence.** Aside from smooth conservative systems, the coexistence phenomenon has been discovered for other classes of dynamical systems (see [37]). In particular, for billiards the coexistence of “elliptic islands” and “chaotic sea” has been shown to be present in *Bunimovich mushrooms* [6]. However, this case differs substantially from the smooth case due to the presence of singularities.

We emphasize that Theorem 1.3 requires some delicate techniques from the theory of nonuniform and partial hyperbolicity. In fact, Fayad [18] constructed an example of a diffeomorphism exhibiting a weaker version of essential coexistence phenomenon: *some but not all* Lyapunov exponents are zero in the regular region. Let us briefly outline his construction. By the conjugation method introduced in [2,19], given $\varepsilon > 0$, one can construct a C^∞ area preserving diffeomorphism T of $\mathbb{T}^2$ such that

(1) T is arbitrarily close to the identity in the C^k topology for every $k > 0$;
(2) T is topologically transitive;
(3) T is not ergodic; moreover, T has a closed invariant subset K of Lebesgue measure greater than $1-\varepsilon$ and such that $K \subset \mathbb{T}^2 \backslash [0,\varepsilon]^2$.

Fix an Anosov automorphism A of $\mathbb{T}^2$. Using various perturbation techniques in partial hyperbolicity theory, one can construct a C^∞ partially hyperbolic volume preserving diffeomorphism P of the 4-dimensional torus $\mathbb{T}^4$ such that

(1) P is arbitrarily close to $A \times T$ and hence, $A \times \mathrm{Id}$ in the C^1 topology;
(2) P has an open and dense ergodic component $\mathcal{U}$ and the central Lyapunov exponents are strictly negative for almost every point in $\mathcal{U}$;
(3) the closed set $\mathcal{K} = \mathbb{T}^2 \times K \subset \mathcal{U}^c$ is invariant under P and has measure greater than $1-\varepsilon$; $P|_{\mathcal{K}} = (A \times T)|_{\mathcal{K}}$; almost every point in $\mathcal{K}$ has one strictly positive Lyapunov exponent, one strictly negative and two equal to zero.

We stress that ensuring that *all* Lyapunov exponents are zero on the exceptional set $\mathcal{U}^c$ of positive measure is a substantially more difficult problem, which requires a completely different set of techniques. The matter is that if all Lyapunov exponents in $\mathcal{U}^c$ are zero, then a typical trajectory that originates in $\mathcal{U}$ will spend a long time in the vicinity of $\mathcal{U}^c$ where contraction and expansion rates are very small. This should be compensated by even longer periods of time that the trajectory should spend away from $\mathcal{U}^c$ thus gaining sufficient contraction and expansion and ensuring nonzero Lyapunov exponents.

The coexistence phenomenon has been numerically observed in various examples in smooth conservative dynamics, most of which come from simple systems of difference or differential equations in small dimension (see [37]).

1.4. **Coexistence phenomenon for other classes of dynamical systems.** Theorem 1.3 can serve as ground for further study of essential coexistence phenomenon for other classes of dynamical systems:

(1) *Symplectic diffeomorphisms*: note that area preserving surface diffeomorphisms, including the standard maps, Przytycki example (1.1), etc. are 2-dimensional symplectic diffeomorphisms. The coexistence phenomenon of first or second types has been studied to certain degree but for symplectic diffeomorphisms in higher dimensions the problem remain completely open and seemingly much more difficult.
(2) *Hamiltonian flows*: there is an example by Donnay and Liverani [15] of a particle moving in a special potential field on the 2-torus which demonstrates an essential coexistence of the second type – the system has positive metric entropy but is not ergodic and the chaotic sea is not dense.
(3) *Geodesic flows on compact Riemannian manifolds*: Donnay [14] constructed an example of a surface on which the geodesic flow exhibits a coexistence phenomenon of the second type. It is obtained by inserting a light-bulb cap into a negative curved surface. In this example the set of geodesics, which are trapped in the cap, is invariant, has positive volume and almost every point in this set has zero Lyapunov exponents. Since it has non-empty interior, the stochastic sea (the set of geodesics that leave the cap) is not dense.

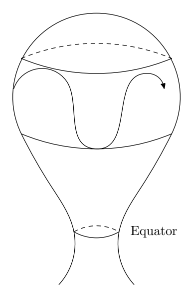

Figure 2. Light-bulb cap

Although the essential coexistence is not yet achieved in either Hamiltonian systems or geodesic flows, it has been established in the category of volume preserving flows by extending Theorem 1.3 to the continuous-time case (see [9]).

Theorem 1.4. *There exists a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 5 and a C^∞ flow $h^t : \mathcal{M} \to \mathcal{M}$ such that*

(1) h^t preserves the Riemannian volume m on $\mathcal{M}$;
(2) $h^t(t \neq 0)$ has nonzero Lyapunov exponents (except for the exponent in the flow direction) almost everywhere on an open, dense and connected subset $\mathcal{U} \subset \mathcal{M}$; moreover, $h^t|\mathcal{U}$ is an ergodic flow;
(3) the complement $\mathcal{U}^c$ has positive volume and is a union of 3-dimensional invariant submanifolds. h^t is a non-identity linear flow with Diophantine frequency vector on each invariant submanifold and h^t has zero Lyapunov exponents on $\mathcal{U}^c$.

We stress that each 3-dimensional invariant submanifold, i.e., $\mathcal{N} \times \{y\}$ for $y \in C$, is in turn a union of 2-dimensional invariant tori on which h^t is a non-identity linear flow with Diophantine frequency vector. This fact makes the construction of the flow nontrivial. We will describe the construction of the flow h^t in Section 4.

2. Pointwise partial hyperbolicity and Lyapunov exponents

One of the principle elements of the constructions in Theorems 1.3 and 1.4 is the notion of pointwise partial hyperbolicity on open sets.

Let f be a diffeomorphism of a compact smooth Riemannian manifold $\mathcal{M}$ and $\mathcal{S} \subset \mathcal{M}$ an f-invariant open subset. We say that f is *pointwise partially hyperbolic on* $\mathcal{S}$ if for every $x \in \mathcal{S}$ the tangent space at x admits an invariant splitting

$$T_x\mathcal{M} = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

into *strongly stable* $E^s(x) = E^s_f(x)$, *central* $E^c(x) = E^c_f(x)$, and *strongly unstable* $E^u(x) = E^u_f(x)$ subspaces. More precisely, there are continuous positive functions $\lambda(x) < \lambda'(x) \leq 1 \leq \mu'(x) < \mu(x)$, $x \in \mathcal{S}$ such that

$$\begin{aligned} \|dfv\| &\leq \lambda(x)\,\|v\|, && v \in E^s(x), \\ \lambda'(x)\,\|v\| \leq \|dfv\| &\leq \mu'(x)\,\|v\|, && v \in E^c(x), \\ \mu(x)\,\|v\| \leq \|dfv\|, & && v \in E^u(x). \end{aligned}$$

Diffeomorphisms that are pointwise partially hyperbolic on the whole manifold $\mathcal{M}$ were introduced in [7]. They have properties that are similar to those of uniformly partially hyperbolic systems:

- (P1) Strongly stable and unstable subspaces E^s and E^u are integrable to continuous strongly stable and unstable foliations W^s and W^u respectively with smooth leaves and these foliations are transverse;
- (P2) Strongly stable and unstable foliations are absolutely continuous;[3]
- (P3) Lyapunov exponents along stable (unstable) subspaces are negative (positive);
- (P4) Any sufficiently small perturbation of a pointwise partially hyperbolic diffeomorphism is also pointwise partially hyperbolic.

These properties may fail if we consider pointwise partially hyperbolic diffeomorphisms on proper open subsets $\mathcal{S} \subset \mathcal{M}$ thus providing one of the major obstacles for our construction. To overcome this difficulty we introduce a special kind of perturbations.

Definition 2.1. *Given a diffeomorphism f that is pointwise partially hyperbolic on an open set $\mathcal{S}$, we call its small perturbation g in the C^1 topology* gentle *if there exists an open set $\mathcal{U} \subset \mathcal{S}$ such that*

(1) $\bar{\mathcal{U}} \subset \mathcal{S}$;
(2) $\mathcal{U}$ is invariant under both f and g;
(3) $f|\mathcal{U}^c = g|\mathcal{U}^c$.

It is easy to see that a gentle perturbation of a diffeomorphism g that is pointwise partially hyperbolic on an open set $\mathcal{S}$ is also pointwise partially hyperbolic on $\mathcal{S}$.

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[3]We say that a foliation W on $\mathcal{S}$ is *absolutely continuous* if for almost every $x \in \mathcal{S}$ there is a neighborhood $B(x, q(x))$ such that for almost every $y \in B(x, q(x))$ the conditional measure generated on the local leave $V(y)$ by volume m is absolutely continuous with respect to the leaf volume $m_{V(y)}$ on $V(y)$.

If the center distribution E^c_f is integrable to a smooth center foliation W^c_f, then f and its small gentle perturbation are *dynamically coherent*[4] (see [25, 34]).

Let f be a diffeomorphism that is pointwise partially hyperbolic on an open set $\mathcal{S}$ and has Property (P1). Let also $\widetilde{\mathcal{S}} \subset \mathcal{S}$ be an open subset.

Definition 2.2. *We say that f has the* accessibility property *(with respect to $\widetilde{\mathcal{S}}$ and $\mathcal{S}$) if any two points $z, z' \in \widetilde{\mathcal{S}}$ are* accessible *via the strongly stable and unstable foliations W^s and W^u, that is,*

1. *there exists a collection of points $z_1, \dots, z_n \in \mathcal{S}$ such that $x = z_1$, $y = z_n$ and $z_k \in V^i(z_{k-1})$ for $i = s$ or u and $k = 2, \dots, n$;*
2. *the points z_{k-1} and z_k can be connected by a smooth curve $\gamma_k \subset V^i(z_{k-1})$ for $i = s$ or u and $k = 2, \dots, n$.* [5]

The collection of such points z_k and curves γ_k is called the (u, s)-*path* connecting z and z'.

We denote by $\lambda_i(x) = \lambda_i(x, f)$, $i = 1, \dots, \dim \mathcal{M}$, the values of the Lyapunov exponents at x, counted with their multiplicities and arranged in the decreasing order. We also denote by

$$L_k(f) := \int_{\mathcal{M}} \sum_{i=1}^{k} \lambda_i(x, f)\, dm(z), \tag{2.1}$$

the *k-th average Lyapunov exponent* of f with respect to volume m. Note that $L_k(\cdot)$ is upper-semicontinuous in the space of C^1 diffeomorphisms on $\mathcal{M}$.

Consider a volume preserving C^2 diffeomorphisms f of a compact smooth manifold $\mathcal{M}$ that is pointwise partially hyperbolic on an open set $\mathcal{S}$.

Definition 2.3. *We say that f has* positive central exponents *if there is an invariant set $\mathcal{A} \subset \mathcal{S}$ of positive volume such that for every $x \in \mathcal{A}$ and every $v \in E^c(x)$ the Lyapunov exponent $\lambda(x, v) > 0$.*

The following result plays an important role in the proof of Theorem 1.3.[6]

Theorem 2.4. *Assume that the following conditions hold:*

1. *f has strongly stable and unstable foliations W^s and W^u on $\mathcal{S}$;*
2. *the foliations W^s and W^u are absolutely continuous;*
3. *f has the accessibility property via the foliations W^s and W^u;*
4. *f has positive Lyapunov exponents in the strongly unstable directions and negative Lyapunov exponents in the strongly stable directions almost everywhere;*
5. *f has positive central exponents.*

[4]A diffeomorphism f that is pointwise partially hyperbolic on an open set $\mathcal{S}$ is called *dynamically coherent* if the subbundles $E^{cu} = E^c \oplus E^u$, E^c, and $E^{cs} = E^c \oplus E^s$ are integrable to continuous foliations with smooth leaves W^{cu}, W^c and W^{cs}, called respectively the *center-unstable*, *center* and *center-stable foliations*. Furthermore, the foliations W^c and W^u are subfoliations of W^{cu}, while W^c and W^s are subfoliations of W^{cs}.

[5]We stress that $V^i(z_{k-1})$ is the *local* leaf of W^i at z_{k-1}. In particular, the length of the curve γ_k (the *leg* of the path) does not exceed the size of $V^i(z_{k-1})$.

[6]In the case when f is uniformly partially hyperbolic on the whole manifold $\mathcal{M}$, has positive central exponents and the accessibility property this result was proved in [3].

Then f has positive central exponents at almost every point $x \in \mathcal{S}$, $f|\mathcal{S}$ is ergodic and indeed, is a Bernoulli diffeomorphism.

All the above notions and results can be extended to the continuous-time case, for example, a flow f^t is pointwise partially hyperbolic on an open subset $\mathcal{S}$ if its time-1 map f^1 is pointwise partially hyperbolic on $\mathcal{S}$. We point out two important differences between the desecrete- and continuous-time cases: (1) the Lyapunov exponent in the flow direction, which is part of the central direction, is always zero; and (2) in Theorem 2.4 restated to the case of flows one can occlude that the flow is ergodic but may not be isomorphic to a Bernoulli flow.

3. Essential coexistence: the discrete-time case

We outline the construction in Theorem 1.3. It starts with a C^∞ volume preserving diffeomorphism T of a compact smooth 5-dimensional manifold $\mathcal{M}$ given as follows. Let A be an Anosov automorphism of the torus $X = \mathbb{T}^2$ and T^t the suspension flow over A with a constant roof function. The flow acts on the suspension manifold

$$\mathcal{N} = \{(x,t) : x \in X,\, \tau \in [0,1]\}/\sim,$$

where "$\sim$" is the identification $(x,1) = (Ax,0)$.

Set $Y = \mathbb{T}^2$ and $\mathcal{M} = \mathcal{N} \times Y$. We endow $\mathcal{M}$ with the product metric and denote by m its Riemannian volume. This is the desired manifold.

Next we choose a Cantor set $C \subset Y$ of positive area whose complement $U = Y \setminus C$ is an open connected subset and we also choose a C^∞ function $\kappa : Y \to \mathbb{R}$, which vanishes on C, is positive on U and is constant on an open subset U_0 with $\overline{U}_0 \subset U$.

The set $\mathcal{U} = \mathcal{N} \times U$ is open, dense and of positive but not full volume. This is the desired open set $\mathcal{U}$ and its complement $\mathcal{U}^c = \mathcal{N} \times C$ is the desired Cantor set of positive volume.

Define the "start-up" map $T \colon \mathcal{M} \to \mathcal{M}$ by

$$T((x,\tau),y) = (T^{\kappa(y)}(x,\tau),y),$$

where $(x,\tau) \in \mathcal{N}$ and $y \in Y$. It is easy to see that the map T is a C^∞ volume preserving diffeomorphism of $\mathcal{M}$ such that

- (T1) given $\delta > 0$, one can choose the function κ such that $\|T - \mathrm{Id}\|_{C^1} \le \delta$;
- (T2) T preserves the fibers $\mathcal{N} \times \{y\}$;
- (T3) $T|\mathcal{U}^c = \mathrm{Id}$ and $dT_z = \mathrm{Id}$ for any $z \in \mathcal{U}^c$; in particular, the Lyapunov exponents of $T|\mathcal{U}^c$ are all zero;
- (T4) the map T is pointwise partially hyperbolic on an open set $\mathcal{U}$ with one-dimensional strongly stable $E^s_T(z)$, one-dimensional strongly unstable $E^u_T(z)$ and three-dimensional center $E^c_T(z)$ subspaces;
- (T5) for every $z \in \mathcal{U}$ the map T has one negative, one positive and three zero Lyapunov exponents.

In our construction of the desired map P we follow the perturbation scheme

$$T \to Q \to P,$$

where both maps Q and P are pointwise partially hyperbolic on $\mathcal{U}$ with one-dimensional stable, one-dimensional unstable, and three-dimensional central subspaces. The map Q is a gentle perturbation of T and hence has Properties (P1)–(P3). We shall ensure that Q has positive central exponents on a subset of positive measure. Although the map P is *not* a gentle perturbation of Q it is obtained in a way that allow us to maintain Properties (P1)–(P3). We shall ensure that P has the accessibility property on $\mathcal{U}$ which allows us to apply Theorem 2.4.

3.1. **Construction of the map Q.** The map Q is a gentle perturbation of T and is concentrated in the open set $\mathcal{N} \times U_0 \subset \mathcal{U}$ which is "far away" from the Cantor set $\mathcal{N} \times C$. The construction of the map Q is described by the following proposition.

Proposition 3.1. *Given $\delta > 0$, there is a C^∞ volume preserving diffeomorphism $Q : \mathcal{M} \to \mathcal{M}$ such that*

(1) $\|Q - T\|_{C^1} \leq \delta$;
(2) $Q = T$ *on the set* $\mathcal{N} \times (Y \setminus U_0)$ *and* Q *preserves each fiber* $\mathcal{N} \times \{y\}$ *for* $y \notin U_0$; *in particular,* Q *is a gentle perturbation of* T *and hence, it is pointwise partially hyperbolic on* $\mathcal{U}$ *and has Properties (P1)–(P3);*
(3) there is an invariant set $\mathcal{A} \subset \mathcal{U}$ *of positive volume such that for every* $z \in \mathcal{A}$,

$$\lambda^u(z, Q) = \lambda_1(z, Q) > \lambda_2(z, Q) > \lambda_3(z, Q) > \lambda_4(z, Q) > 0$$

while $\lambda_5(z, Q) = \lambda^s(z, Q) < 0$.

The last property implies that Q has distinct average Lyapunov exponents, i.e.,

$$L_1(Q) < L_2(Q) < L_3(Q) < L_4(Q) = L_4(T)$$

and $L_5(Q) = 0$ where $L_k(\cdot)$ is the k-th average Lyapunov exponent defined in (2.1). While individual values of the Lyapunov exponents may vary widely under small perturbations, average Lyapunov exponents are upper semi-continuous and thus can be controlled.

The construction of the map Q exploits some methods from [5, 12, 13, 28, 35] but requires substantial modifications and new arguments due to nonuniform hyperbolicity of the map T. Note that the restriction $Q|\mathcal{U}$ is not ergodic.

To construct the map Q, we follow the perturbation scheme

$$T \to S = h_S \circ T \to R = h_R \circ S \to Q = h_Q \circ R,$$

where h_S, h_R and h_Q are gentle perturbations. Observe that

$$L_1(T) = L_2(T) = L_3(T) = L_4(T) > 0.$$

The perturbations h_S, h_R and h_Q are designed to achieve that

$$L_1(S) < L_2(S)$$

then

$$L_1(R) < L_2(R) < L_3(R)$$

and finally,

$$L_1(Q) < L_2(Q) < L_3(Q) < L_4(Q).$$

The perturbations h_K, $k = S, R, Q$, are concentrated on pairwise disjoint sets Ω_k, $\|h_k - \mathrm{Id}\| \leq \delta$ and $h_k = \mathrm{Id}$ outside the set $\Omega = \Omega_S \cup \Omega_R \cup \Omega_Q$. Furthermore, $\Omega \subset \mathcal{N} \times U_0$.

The first perturbation h_S creates positive Lyapunov exponent in the τ-direction - the direction of the special flow on a subset $\mathcal{A} \subset \mathcal{U}$ of positive volume. This can be achieved by applying a small rotation in the $E^{u\tau}$-plane at every point in a small neighborhood Ω_S of some point $z_0 \in \mathcal{N} \times U_0$. The idea originated in the work of Shub-Wilkinson [35] and of Dolgopyat-Hu-Pesin [12].

More precisely, we choose the coordinate system (s, u, τ, a, b) around a given point $z_0 \in \mathcal{N} \times U_0$ and the set $\Omega_S = \{(s, u, \tau, a, b) : |s| \le \varepsilon, u^2 + \tau^2 \le \varepsilon^2, |a|,\ |b| \le \alpha\}$, and switch to the cylindrical coordinates (r, θ, s, a, b) where $u = r\cos\theta$ and $\tau = r\sin\theta$. For small $\sigma > 0$ set

$$h_S(r, \theta, s, a, b) = (r, \theta + \sigma\alpha^2\varepsilon^2\psi(\frac{r}{\varepsilon})\psi(\frac{s}{\varepsilon})\psi(\frac{|a|}{\alpha})\psi(\frac{|b|}{\alpha}), s, a, b)$$

if the point $(r, \theta, s, a, b) \in \Omega_S$ and set $h_S = \mathrm{Id}$ otherwise. Here ψ is a C^∞ function ψ on $\mathbb{R}^+$ such that $\psi(r)$ is constant on $[0, 0.9]$, $\psi(r) > 0$ for $r \in [0, 1]$ and $\psi(r) = 0$ for $r \ge 1$.

The second perturbation h_R creates positive Lyapunov exponent in the a-direction of Y on a subset $\mathcal{B} \subset \mathcal{A}$ of positive volume leaving the Lyapunov exponent in the τ-direction still positive. This map is a composition of very small rotations in the τa-plane along long segments of non-periodic orbits of the map S so that the total rotation is $\frac{\pi}{2}$. In this way, the map $R = h_R \circ S$ interchanges the τ- and a-directions making the average Lyapunov exponents along these directions non-zero. The idea goes back to some work of Mané (unpublished) and of Dolgopyat-Pesin [13].

To explain this idea we pick $\lambda > 0$ such that $\lambda_2(z, S) \ge \lambda$ for all $z \in \mathcal{A}$ and consider a segment of trajectory at a typical point z of length $2k+m$ where both k and m are very large and k is much larger than m. Given a vector $v \in E_S^{u\tau a}(z)$, if v is close to the $E_S^{u\tau}$-subspace, then the length of the $u\tau$-component under the first k iterations becomes at least about λ^{-k} times longer than the length of v. Since dR does not contract vectors in the $E_S^{u\tau a}$-subspace very much during the remaining $m + k$ iterations, the length of the $u\tau$-component stays about the same. On the other hand, if v is close to the E_S^a-subspace, then the length of the a-component of v does not change under the first k iterations. During the next m iterations we apply very small rotations in the τa-plane so that the vector $dR^k v$ is rotated by $\pi/2$ degree into the E_S^τ-subspace. During the next k iterations the length of the vector becomes at least about λ^{-k} times longer. It follows that every vector in $E^{u\tau a}(z)$ expands by about λ^{-k} times under dR^{2k+m}.

The last perturbation h_Q creates positive Lyapunov exponent in the b-direction in Y on a subset $\mathcal{C} \subset \mathcal{B}$ of positive volume leaving the Lyapunov exponents in the τ- and a-directions still positive. It is constructed in a way similar to the construction of the map h_R. Thus the map Q has positive Lyapunov exponents in the three-dimensional central direction.

3.2. **Construction of the map P.** The desired map P is obtained as a limit of some specially chosen gentle perturbations of T. It is not a gentle perturbation on its own and therefore, additional arguments are needed to establish Properties (P1)–(P3) for P.

As a small perturbation of Q, the map P still has positive Lyapunov exponents in the central directions on a subset of positive volume and has zero Lyapunov exponents on the Cantor set $\mathcal{U}^c = \mathcal{N} \times C$. Moreover, we shall construct P in such a way that it has the accessibility property on the open set $\mathcal{U}$ via its strongly stable and unstable foliations. In order to do this, we use some techniques developed in [12, 28].

To effect the construction, we regard the 2-torus Y as the square $[0, 8] \times [0, 8]$ whose opposite sides are identified and choose two special collections of squares $Z_j^{(n)}$ and $\tilde{Z}_j^{(n)}$ for $n \ge 1$ and $j = 1, \dots, k_n$ such that they form two open covers of the torus by "slightly overlapping" squares and $\tilde{Z}_j^{(n)} \subset Z_j^{(n)}$. Set

$$U_n = \bigcup_j Z_j^{(n)}, \quad \tilde{U}_n = \bigcup_j \tilde{Z}_j^{(n)}.$$

By this specific construction, we have that $\widetilde{U}_n$ and U_n are connected sets,

$$\overline{U}_0 \subset \widetilde{U}_1, \quad \widetilde{U}_n \subset \overline{\widetilde{U}}_n \subset U_n \subset \overline{U}_n \subset \widetilde{U}_{n+1}$$

and

$$\bigcup_{n\geq 1} U_n = \bigcup_{n\geq 1} \widetilde{U}_n = U.$$

Then we set

$$\mathcal{U}_n = \mathcal{N} \times U_n, \quad \widetilde{\mathcal{U}}_n = \mathcal{N} \times \widetilde{U}_n.$$

The sets $\mathcal{U}_n$ are nested and exhaust the open set $\mathcal{U}$.

We then construct a sequence of diffeomorphisms $\{P_n\}$ each acting on the corresponding set $\mathcal{U}_n$, whose limit is the desired map P. More precisely, in the 2-torus X we choose quadruples of periodic points $\{q^i, p^i_\tau, p^i_a, p^i_b\}$, $i = 1, \dots, 8$ for the linear hyperbolic automorphism A and small neighborhoods Π^i_ℓ of q^i containing p^i_ℓ, $\ell = \tau, a, b$. One can associate $i = i(n, j)$ for each square $Z^{(n)}_j$ and define the domains

$$\Omega^\ell_{n,j} = \{z = (x, \tau, a, b) : \ x \in \Pi^i_\ell, \ |\tau| \leq \epsilon, \ (a, b) \in Z^{(n)}_j\}.$$

Switching to the coordinate $(u, \ell, *, *, *)$, $\ell = \tau, a, b$, we define the vector fields in $\Omega^\ell_{n,j}$ by

$$X^\ell_{\beta,j,n}(z) = \beta\psi^\ell(s)\big(-\int_0^u \phi^\ell(r)\, dr, \phi^\ell(u), 0, 0, 0\big)$$

for sufficiently small $\beta = \beta_n > 0$, where the C^∞ functions ϕ^ℓ and ψ^ℓ on $\mathbb{R}$ satisfy:

- $\phi^\ell(r) = \text{const.}$ and $\psi^\ell(r) = \text{const.}$ for $r \in (-r^\ell_0, r^\ell_0)$;
- $\phi^\ell(r) = 0$ and $\psi^\ell(r) = 0$ for $|r| \geq r^\ell_0$;
- $\int_0^{\pm r^\ell_0} \phi^\ell(\tau) d\tau = 0$ and $\psi^\ell(x) > 0$ for any $|x| < r^\ell_0$;
- $\|\phi^\ell(\cdot)\|_{C^n} < 1$ and $\|\psi^\ell(\cdot)\|_{C^n} < 1$.

Let $h^\ell_{\beta,j,n}$ be the time-1 map of the flow generated by the vector field $X^\ell_{\beta,j,n}$ and set $h^\ell_{\beta,j,n} = \text{Id}$ on the complement of $\Omega^\ell_{n,j}$. Clearly, $X^\ell_{\beta,j,n}$ is divergence free and of class C^∞ and hence, $h^\ell_{\beta,j,n}$ preserves volume and is a C^∞ diffeomorphism. Composing the maps $h^\ell_{\beta,j,n}$ over ℓ and j for each $n \geq 1$, we obtain the desired perturbation $h_n = h_{n,\beta_n}$ so that $P_n = h_n \circ h_{n-1} \circ \dots \circ h_1 \circ Q$. One can show that the maps P_n have the following properties.

Proposition 3.2. *Given $\delta > 0$, there are two sequences of positive numbers $\{\delta_n\}$ and $\{\theta_n\}$ satisfying $\delta_n \leq \delta/2^n$ and $\delta_n \leq d(C, U_n)^2$ and a sequence of C^∞ volume preserving diffeomorphisms $P_n : \mathcal{M} \to \mathcal{M}$ such that for $n \geq 1$:*

(1) $||P_n - P_{n-1}||_{C^n} < \delta_n$;
(2) $P_n(\mathcal{U}_n) = \mathcal{U}_n$, $P_n = T$ on $\mathcal{M} \setminus \mathcal{U}_n$, and $P_n = P_{n-1}$ on $\mathcal{U}_{n-2}$;
(3) P_n is a gentle perturbation of T and hence, is pointwise partially hyperbolic on $\mathcal{U}$ and has two transverse stable and unstable foliations $W^s_{P_n}$ and $W^u_{P_n}$;
(4) each map P_n is accessible via $W^s_{P_n}$ and $W^u_{P_n}$; more precisely, any $z, z' \in \mathcal{U}_n$ can be connected via a $(u, s)_{P_n}$-path in $\widetilde{\mathcal{U}}_n$;
(5) for all $z \in \mathcal{U}_j$, $j = 1, \dots, n$ and $i = u, s, c$,

$$\angle(E^i_{P_{n-1}}(z), E^i_{P_n}(z)) \leq \theta_j/2^{n-j}.$$

We outline the argument allowing to achieve the accessibility property in Statement (4). First, it is not difficult to show that the accessibility property holds if for a certain point $z \in \mathcal{U}_n$ every point in its local central manifold $V^c(z)$ is accessible from z. To this end using the well-known Brin's quadrilateral argument ,one can construct a function from the unit cube in $\mathbb{R}^3$ to $V^c(z)$ such that every point in the image of the function is accessible from z. Furthermore, one can show that this function is continuous, which guarantees that it is onto $V^c(z)$, and hence every point in $V^c(z)$ is accessible from z.

Statements (1) and (2) of Proposition 3.2 imply that the limit $P = \lim_{n\to\infty} P_n$ exists. Indeed, for any $k \geq 1$ and any $n > k$,

$$||P_n - P_{n-1}||_{C^k} \leq ||P_n - P_{n-1}||_{C^n} < \delta/2^n.$$

It follows that P_n converges to P in the C^k topology. Since k is arbitrary, P is a C^∞ diffeomorphism. Clearly, P preserves volume and $\|P - \mathrm{Id}\| \leq \delta$ and the first statement of the theorem follows.

If the sequence δ_n decays sufficiently fast then for any $n > k$, any $z \in \mathcal{U}_k$, and $i = s, u, c$ we have

$$\angle(E^i_{P_n}(z), E^i_{P_k}(z)) \leq \theta_k(1 - \frac{1}{2^k}) < \theta_k.$$

Thus the sequence of subspaces $\{E^i_{P_n}(z)\}$ is Cauchy and hence converges as $n \to \infty$ to a subspace $E^i_P(z)$. These subspaces form an invariant splitting for P at z. Moreover, since $||P_n - P_{n-1}||_{C^n} \leq \delta_n$, letting $\delta_n \to 0$ sufficiently fast as $n \to \infty$ we can ensure that the map P is pointwise partially hyperbolic with one-dimensional strongly stable E^s_P, one-dimensional strongly unstable E^u_P and three-dimensional central E^c_P subspaces. This shows that the limit map P is very flat near the Cantor set $\mathcal{N} \times C$ and thus ensures zero Lyapunov exponents in the directions transversal to the Cantor set.

We use the fact that the strongly stable subspace $E^s_P(z)$ is one-dimensional to show that the Lyapunov exponent $\lambda^s(z, P)$ in the stable direction is negative at almost every point $z \in \mathcal{U}$. Indeed, let $Z \subset \mathcal{U}$ be the set of points at which $\lambda^s(z, P) = 0$. If $m(Z) > 0$ then

$$0 = \int_Z \lambda^s(z, P)\, dm = \int_Z \lim_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} \lambda(P^i(z))$$

$$= \lim_{n\to\infty} \frac{1}{n} \int_Z \sum_{i=0}^{n-1} \log \lambda(P^i(z))\, dm(z) = \int_Z \log \lambda(z)\, dm(z) < 0,$$

where $\lambda(z)$ is the contraction coefficient along $E^s_P(z)$. This contradiction proves the claim. Similarly, one can prove that the Lyapunov exponent $\lambda^u(z, P)$ in the direction $E^u_P(z)$ is positive at almost every point $z \in \mathcal{U}$.

The strongly stable E^s_P and unstable E^u_P subbundles are integrable to invariant strongly stable W^s_P and unstable W^u_P foliations with smooth leaves, which are transverse. Indeed, given $z \in \mathcal{U}$, the size of the local stable manifold $V^s_{P_n}(z)$ through z is larger than a certain number $r > 0$, which is independent of z and n. Hence, the sequence of local manifolds $V^s_{P_n}(z)$ converges in the C^1 topology to a local manifold of size at least r. This manifold is tangent to the stable subspace at z and is a local stable manifold for P at z.

Further, the map P has the accessibility property on $\widetilde{\mathcal{U}}_k$. Since k is arbitrary, we obtain that the map P has the accessibility property on $\mathcal{U}$ via its invariant foliations W^s_P and W^u_P.

To prove that the map P has nonzero central Lyapunov exponents on a set of positive volume we let $c = L_4(Q) - L_3(Q) > 0$. By semicontinuity of $L_k(\cdot)$ with respect to the map, we may take δ so small that $L_3(P) < L_3(Q) + c/2$. Then for all $n \geq 1$,

$$L_4(P_n) = \int_{\mathcal{U}} \log|\det(dP_n|E^{utab}_{P_n}(z))|\,dm$$

$$= \int_{\mathcal{U}} \log|\det(dQ|E^{utab}_{Q}(z))|\,dm = L_4(Q).$$

Since P_n converges to P in the C^1 topology, $L_4(P_n) \to L_4(P)$ as $n \to \infty$ and hence $L_4(P) = L_4(Q)$. It follows that $L_4(P) - L_3(P) \geq c/2 > 0$. Therefore,

$$\int_{\mathcal{U}} \lambda_4(z,P)dm(z) \geq c/2 > 0.$$

It follows that there is a subset $\mathcal{A} \subset \mathcal{U}$ of positive volume such that $\lambda_4(z,P) > 0$ for every $z \in \mathcal{A}$. Hence,

$$\lambda_2(z,P) \geq \lambda_3(z,P) \geq \lambda_4(z,P) > 0.$$

Thus P has positive central exponents at every point in a set of positive volume. Since P is volume preserving, the total sum of the Lyapunov exponents is zero at every point. Therefore, $\lambda_5(z,P) < 0$ for every $z \in \mathcal{A}$.

Since $\delta_n \leq d(C,U_n)^2$, we obtain that $P = \mathrm{Id}$ on the set $\mathcal{N} \times C$ and that $dP_z = \mathrm{Id}$ for all $z \in \mathcal{N} \times C$. In other words, all Lyapunov exponents at every point in the set $\mathcal{N} \times C$ are zero. Since this set has positive volume this completes the proof of the Theorem.

4. Essential Coexistence: the continuous-time case

Now we explain how to obtain a flow with the essential coexistence in Theorem 1.4 by modifying the construction in Section 3. We consider the same compact manifold $\mathcal{M}$ and the open subset $\mathcal{U}$ but we introduce a start-up flow f^t by the formula

$$f^t((x,\tau),y) = ((x + t\alpha(y), \tau + t\kappa(y)), y),$$

where $\kappa : Y \to \mathbb{R}$ is the same function as in Section 3 and $\alpha : Y \to \mathbb{R}^2$ is a C^∞ map such that

(1) α vanishes on the set U_0;
(2) α equals to a constant Diophantine vector on the Cantor set C;
(3) $\sup_{y\in Y} \|\alpha(y)\| \leq \bar{\alpha}$, where $\bar{\alpha}$ is a small positive number.

Clearly, f^t is a C^∞ volume preserving flow of $\mathcal{M}$ satisfying properties similar to (T1)–(T5) in Section 3. Furthermore, each three-dimensional fiber $\mathcal{N} \times \{y\}$, $y \in C$, is a union of teo-dimensional invariant tori $X \times \{\tau\} \times \{y\}$, $\tau \in [0,1]$, on which f^t acts as a linear flow with a Diophantine frequency vector.

We perturb the flow f^t gently by the following scheme

$$f^t \to g^t \to h^t$$

such that both flows g^t and h^t are pointwise partially hyperbolic on $\mathcal{U}$. The flow g^t has positive central Lyapunov exponents on a set of positive volume but is not necessarily ergodic. Then we create the accessibility property for the desired flow h^t.

4.1. **Construction of the flow g^t.** We extend the approach introduced in Subsection 3.1 to the case of flows and obtain the flow g^t as a result of two consecutive perturbations

$$f^t \to \widetilde{g}^t \to g^t$$

such that

$$L_1(\widetilde{g}^t) < L_2(\widetilde{g}^t)$$

then

$$L_1(g^t) < L_2(g^t) < L_3(g^t).$$

Note that $L_3(g^t) = L_4(g^t)$ since the flow direction must have zero exponent. In fact, we apply perturbations of corresponding vector fields

$$\mathcal{X}_f \to \mathcal{X}_{\widetilde{g}} = \mathcal{X}_f + \widetilde{\mathcal{X}}^R \to \mathcal{X}_g = \mathcal{X}_{\widetilde{g}} + \mathcal{X}^R,$$

where $\mathcal{X}_f$ is the vector field that generates the flow f^t, and $\widetilde{\mathcal{X}}^R$ and $\mathcal{X}^R$ are two rotational vector fields supported on pairwise disjoint open subsets $\widetilde{\Omega}_R$ and Ω_R of $\mathcal{N} \times U_0$ respectively such that $\widetilde{\mathcal{X}}_R = 0$ outside $\widetilde{\Omega}_R$, $\mathcal{X}_R = 0$ outside Ω_R and $\|\widetilde{\mathcal{X}}_R\|_{C^1},\ \|\mathcal{X}_R\|_{C^1} < \delta$.

The role of these two vector fields $\widetilde{\mathcal{X}}^R$ and $\mathcal{X}^R$ is similar to that of perturbations h_S, h_R, h_Q in Subsection 3.1. The perturbation via $\widetilde{\mathcal{X}}^R$ produces two positive average Lyapunov exponents for the flow $\widetilde{g}^t$ in the E^{ua}_f subbundle, and then the perturbation via $\mathcal{X}^R$ makes the flow g^t have three positive average Lyapunov exponents in the E^{uab}_f subbundle.

The construction utilizes the following special and crucial feature of the flow f^t: the set

$$\Pi_0 = X \times \{0\} \times U_0$$

is a global cross-section of $f^t|\mathcal{N} \times U_0$, and the time-1 map restricted to Π_0 is exactly the Poincaré return map of f^t to Π_0. Furthermore, we make the construction of vector fields $\widetilde{\mathcal{X}}^R$ and $\mathcal{X}^R$ in such a way that Π_0 is also a global cross-section for both flows $\widetilde{g}^t$ and g^t with the time-1 maps to be the Poincaré return map to Π_0. This fact allows us to apply arguments similar to those in Subsection 3.1 to our flow case by focusing on the time-1 maps.

4.2. **Construction of the flow h^t.** Similar to the construction of the map P in Subsection 3.2, we obtain our desired flow h^t as a limit of properly chosen gentle perturbations h^t_n. We guarantee that h^t is C^1-close to the flow g^t such that h^t still has positive central exponents (except for the flow direction) on a subset of positive volume. Moreover, h^t has the accessibility property on the open set $\mathcal{U}$ via its strongly stable and unstable foliations.

Note that the perturbations in Subsection 3.2 indeed come from the construction of perturbed vector fields. Therefore, we can adjust this construction to the continuous-time case such that the sequence of flows h^t_n satisfy properties similar to those in Proposition 3.2.

There is however an important modification in constructing the perturbations h^t_n: instead of choosing 8 quadruples of periodic points (as in Subsection 3.2), we choose for each square $Z^{(n)}_j$ centered at $y_0 = y_0(n, j)$ a quadruple of periodic points $\{q^{(n,j)}, p^{(n,j)}_\tau, p^{(n,j)}_a, p^{(n,j)}_b\}$ in the 2-torus X for the Anosov affine map $A + \alpha(y_0)/\kappa(y_0)$. Moreover, we can guarantee that the corresponding domains $\Omega^\ell_{n,j}$, $\ell = \tau, a, b$, are pairwise disjoint. In this way, we can obtain accessibility property by applying a small perturbation supported in each domain, which do not interrupt each other.

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