The multifractal analysis of Gibbs measures: Motivation, mathematical foundation, and examples

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We first motivate the study of multifractals. We then present a rigorous mathematical foundation for the multifractal analysis of Gibbs measures invariant under dynamical systems. Finally we effect a complete multifractal analysis for several classes of hyperbolic dynamical systems. © *1997 American Institute of Physics*. [S1054-1500(97)00601-0]

This manuscript has several objectives. We first describe some models in the natural sciences (Richardson's model of turbulence, distribution of galaxies in the universe, and root systems in plant biology) where the fundamental objects of study are multifractals. We then present a rigorous mathematical foundation for the multifractal analysis of measures invariant under dynamical systems with conformal and hyperbolic behavior. We consider measures that are natural from a physical point of view and capture information on the self-similar properties of the distribution of trajectories on strange attractors. Finally, we effect a complete multifractal analysis for conformal expanding maps and Smale's horseshoes.

I. INTRODUCTION

Invariant sets of dynamical systems in general are *not* self-similar in the strict sense. However, part of these sets can sometimes be decomposed into (perhaps innumerable) subsets each supporting a Borel probability measure exhibiting a type of scaling symmetry. This means that the measure admits a group of scale symmetries that reproduces copies of the set (or its significant part of full measure) on arbitrarily small scales (up to a given precision that decreases with the scale). Sets that admit such structure are called *multifractals*. The Hausdorff dimension of each subset can be used to characterize this structure. The detailed analysis of the multifractal structure of a set invariant for a chaotic dynamical system allows one to obtain a more refined description of the chaotic behavior than the description based upon purely stochastic characteristics (e.g., Lyapunov exponents and measure theoretic entropy).

The concept of a multifractal analysis was suggested by a group of physicists in the seminal paper (HJKPS) where the authors attempted to understand the multiscaling behavior of physical measures on strange attractors, diffusion-limited aggregates, etc. The multifractal analysis of measures on limit sets has since become a popular interdisciplinary subject of study—a search of several electronic databases showed that there are now hundreds of related papers in physical, engineering, biological, and mathematical literature. There are currently 152 papers in the MathSci database with multifractal or multiscaling in the title.

Below are three diverse areas of current investigation using the multifractal/multiscaling analysis that we find particularly fascinating. We believe that understanding the underlying multifractal structure will play an important role in solving each of these problems.

(1) The first application is the study of turbulence [J]. According to Richardson's description of turbulence, there is a cascade of transfers of energy from large down to small scales. The cascade is hierarchical in the sense that a disturbance on a certain scale receives energy from a larger scale and transfers it to smaller scale disturbances. At the end of the cascade the smallest disturbances are characterized by very large velocity gradients because the conversion of kinetic energy into heat is strongly localized.

Under the assumptions that the rate of transfer of energy is constant both in space and in the steps of the energy cascade, one can obtain the famous Kolmogorov scaling law for velocity differences $\langle |v(x+h)-v(x)|^q\rangle \approx h^{-q/3}$, where $\langle\cdot\rangle$ denotes spacial average. The case $q=1$ is frequently referred to as Kolmogorov's 1/3 law of turbulence.

During the last decade there has been much experimental and numerical evidence showing that strong fluctuations of the energy transfer and dissipation are present, a phenomenon called intermittency, and that the Kolmogorov law seems to break down for large values of q. In particular, investigators observed that the set of high vorticity is a thread-like fractal set that is definitely not homogeneous in space. Several authors have attempted to analyze the multifractal structure of the energy dissipation by studying the Rényi spectrum of the energy dissipation density and have proposed (phenomenological) corrections to the Kolmogorov law utilizing dimension-like characteristics.

(2) The second application is the study of the distribution of galaxies and clusters of galaxies in the universe [MPBC, Spi]. One of the key problems in modern cosmology is understanding how the spatial clustering of objects such as galaxies can provide clues about the evolution of the primordial density inhomogeneities under the action of gravitational instability. Since clusters are considered multifractals, some characteristics of the multifractal structure (such as correlation dimension, information dimension, etc.) may be useful in a physical theory to describe the distribution of clusters.

(3) The third application is (plant) biology. The relationship between form and function, particularly resource capture, is one of the central problems in organismic biology.

Root systems are a particularly interesting and important object of study. Roots explore a solid medium for relatively immobile resources such as phosphorous. It is hoped that a multifractal analysis of the metabolic activity of root structures will provide insights on the ability of root systems to efficiently forage in time and space for soil resources [LW].

The first rigorous multifractal analysis of dynamical systems was carried out in [CLP] for a special class of measures invariant under some one-dimensional Markov maps. Later Lopes [Lo] analyzed the multifractal properties of the measure of maximal entropy for a hyperbolic Julia set. In 1994 Simpelaere [Si] effected a multifractal analysis for Gibbs measures of Axiom A surface diffeomorphisms. In this paper we give an alternate proof of Simpelaere's result using the methods developed in [PW3] (see Sec. III).

Our definition of multifractal analysis is faithful to the concepts in [HJKPS] and other articles in physical literature. In addition, our work places these notions onto a solid mathematical foundation. The two major components of the multifractal analysis are the Hentschel–Procaccia (HP) spectrum for dimensions (which should be shown to coincide with the Rényi spectrum for dimensions) and the $f(\alpha)$-spectrum for dimensions (see the descriptions below). The multifractal analysis unifies these two spectra via the Legendre transform (see Appendix B for the definition of Legendre transform). Once the Legendre transform relation between the two spectra is established, one can compute the delicate and seemingly intractable $f(\alpha)$-spectrum through the Rényi or HP spectrum, which is completely determined by the statistics of a single generic trajectory.

There are many papers in the multifractal literature that treat only one of these two components. In a number of papers the pointwise dimension, and thus the $f(\alpha)$-spectrum for dimensions, are studied not with respect to the *natural* metric, but only with respect to the *symbolic metric* [Ra]. This *symbolic pointwise dimension* is, *a priori*, just an intermediary object and is not physically meaningful. In some cases the symbolic pointwise dimension coincides with the usual pointwise dimension, but this is a highly nontrivial result (see Theorems II.4 and III.2). In addition, most authors restrict their analysis to Bernoulli measures or self-similar measures and do not include measures of actual physical interest, like the Bowen–Ruelle–Sinai (BRS) measures on hyperbolic attractors and repellers (or general Gibbs measures).

Let us say a few more words for motivating a mathematical foundation of the multifractal analysis. Let g: $M\to M$ be a diffeomorphism of a smooth Riemannian manifold M and $\Lambda\subset M$ a compact hyperbolic attractor for g. This means that (i) the set Λ carries a (uniformly) hyperbolic structure that is generated by the stable and unstable subspaces at every point $x\in\Lambda$, and (ii) the set Λ is an attractor, i.e., there exists an open neighborhood $\mathcal{U}$ of Λ (the basic of the attractor) such that $f(^{-}\mathcal{U})\subset\mathcal{U}$ and $\Lambda=\cap_{n=0}^{\infty}f^{n}(\mathcal{U})$. For simplicity, we assume that g is topologically mixing on Λ [i.e., given any two open sets U, $V\subset\Lambda$, there exists $N\in\mathbb{N}$ such that for all $n\geqslant N$ we have $g^{n}(U)\cap V\neq O$]. In [B], Bowen showed that the evolution of the Lebesgue measure

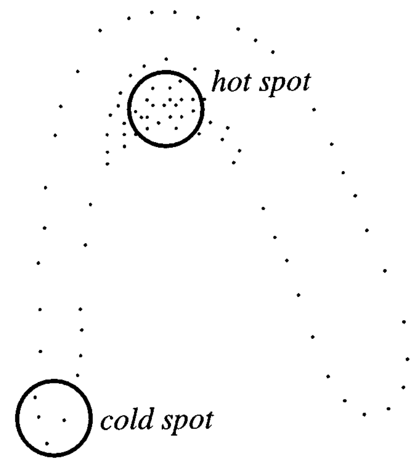

FIG. 1. Hot and cold spots on an attractor.

in the basin $\mathcal{U}$ converges to the BRS measure. From the physical point of view, this is the *natural* measure on the attractor since it describes the orbit distribution of generic points in the basin. This distribution is not uniform, and as computer pictures show, there exist spots of high and low density of visits sometimes called *hot* and *cold* spots. See Fig. 1.

This phenomenon also has been observed for a more general class of attractors (hyperbolic attractors with singularities), which includes the Lorenz attractor, the Lozi attractor, etc. Attempts to analyze this measure in computer simulations are based on partitioning the basin into a very fine grid and estimating the measure of each box by the frequency with which a typical orbit visits it.

An approach to encoding all this data was suggested in [HJKPS] where the authors utilized the *Rényi spectrum for dimensions*, defined as follows. Consider a partition of the attractor by *a grid* of mesh size r, i.e., each partition element contains a ball of radius $\frac{1}{2}r$ and is contained in a concentric ball of radius r. Given a family of grids parametrized by r, define

$$R_{\nu}(q)=\frac{1}{1-q}\lim_{r\to 0}\frac{\log\sum_{i=1}^{N(r)}\nu(C_r^i)^q}{\log r},$$

provided the limit exists (see [T, V]), where ν is a probability distribution in the basin of the attractor and $N(r)$ is the number of partition elements C_r^i of the grid with $\nu(C_r^i)>0$. *A priori*, the limit may depend on the family of grids. We will show that for a large class of measures, called diametrically regular measures, the limit is independent of the family of grids. The result is also true if the number $\frac{1}{2}$ in the definition of a grid is replaced by any positive number.

Another approach, which seems to be experimentally and numerically the most accessible, involves the study of correlations of the distributions of q-tuples along a typical orbit for $q=2,3,\ldots$. This notion was introduced in [G, GHP]. Let g: $X\to X$ be a map on a metric space (X,ρ) preserving a Borel probability measure ν. We set

$$C(x,q,r,n)=\frac{1}{n^q}\,\text{card}\{(i_1\ldots i_q)|\rho(g^{i_j}x,g^{i_k}x)\leqslant r \quad \text{for all } 0\leqslant i_j\leqslant i_k<n\}.$$

We define the *correlation dimension* of order q by

$$C_q(x)=\frac{1}{1-q}\lim_{r\to 0}\lim_{n\to\infty}\frac{\log C(x,q,r,n)}{\log r}$$

provided the limits exist. If ν is ergodic, it was shown in [Pe1] (see also [PT]) that for ν almost every x

$$\lim_{n\to\infty}C(x,q,r,n)=\int_X \nu(B(y,r))^{q-1}d\nu(y),$$

where $B(y,r)$ denotes the ball of radius r centered at the point y. Clearly this limit depends on the metric ρ on X and the ergodic measure ν, but not explicitly on the dynamical system. Thus, for $q=2,3,\ldots$

$$C_q(x)=\frac{1}{1-q}\lim_{r\to 0}\frac{\log \int_X \nu(B(y,r))^{q-1}d\nu(y)}{\log r},$$

provided the limit exists. We stress that in general, one does not expect this limit to exist. In [PT], the authors constructed an example of a continuous map of an interval that preserves a measure absolutely continuous with respect to the Lebesgue measure, for which the above limit does not exist for almost every x in an arbitrarily large interval in q. Combining this with results in [K] one can construct a diffeomorphism of the two-torus, preserving an ergodic measure that is absolutely continuous with respect to the Lebesgue measure, having positive topological entropy, and for which the above limit does not exist for almost every x in an arbitrarily large interval in q. In [PW2] the authors show that this limit exists for a broad class of measures including Gibbs measures on conformal repellers. The limit also exists for Gibbs measures for Axiom A surface diffeomorphisms.

The natural extension of the correlation dimension of order $q=2,3,\ldots$ to all real values $q>1$ was introduced by Hentschel and Procaccia in [HP]. Let ν be a Borel probability measure on a metric space (X,ρ). For $q>1$ we define the HP *spectrum for dimensions* by

$$\text{HP}_\nu(q)=\frac{1}{1-q}\lim_{r\to 0}\frac{\log \int_X \nu(B(y,r))^{q-1}d\nu(y)}{\log r}$$

provided the limit exists.

For an arbitrary Borel probability measure ν, the HP spectrum is not *a priori* defined for $q\leqslant 1$. One problem is that the measure of some balls may be zero. If all balls have positive measure (as in the case of Gibbs measures on repellers and hyperbolic sets), the definition of the HP spectrum for all $q\neq 1$ makes formal sense although the integral may be infinite. The Rényi spectrum was a precursor to the HP spectrum where one replaces the coverings by partitions. We believe that a complete multifractal analysis must include showing that these two spectra agree.

We work with a class of measures that strongly encode the metric structure of the underlying metric space. A measure ν is called *diametrically regular* or a *Federer measure* [Fe] if for a given $A>1$ there exists $K>0$ such that for any sufficiently small $r>0$ and every x we have

$$\nu(B(x,Ar))\leqslant K\nu(B(x,r)). \tag{DR}$$

It is easy to see that if (DR) holds for some number A then it also holds for all positive numbers A. In the harmonic analysis literature such a measure is sometimes called a *doubling measure*. In [PW3] we show that Gibbs measures concentrated on conformal repellers are diametrically regular. This fact plays a crucial role in our multifractal analysis.

In [Pe2], the second author showed that if ν is diametrically regular, then for any $q>1$

$$\text{HP}_\nu(q)=\frac{1}{1-q}\lim_{r\to 0}\frac{\log \inf_{\mathfrak{V}_r}\Sigma_{B\in\mathfrak{V}_r}\nu(B)^q}{\log r},$$

where the infimum is taken over all covers $\mathfrak{V}_r$ of X by balls B of radius r, *provided the limit exists*. We will use this definition of HP spectrum for dimensions in the paper. Also in [Pe2], Pesin showed that the Rényi spectrum coincides with the HP spectrum for diametrically regular measures. In general, even *good* measures may not be diametrically regular (see [Pe2]).

The HP spectrum of dimensions $\text{HP}_\nu(q)$ is not *a priori* defined for $q=1$. It is believed that in *good* cases, $\lim_{q\to 1+}\text{HP}_\nu(q)=I(\nu)$, where $I(\nu)$ is the information dimension (see Remark 3 in Sec. II). It immediately follows from our analysis that this conjecture is true for Gibbs measures for conformal repellers and Axiom A surface diffeomorphisms.

We now turn to the second ingredient in our multifractal analysis and define the $f_\nu(\alpha)$-spectrum for dimensions. To motivate this, let us briefly return to the study of hot and cold spots on an attractor. Cover the attractor by a uniform grid of mesh size r. Let p_i be the average number of visits of a *typical* orbit to a given box B_i of the grid, i.e., $p_i=\mu(B_i)$, where μ is a natural measure. The collection of numbers $\{p_i\}$ determine the distribution of hot and cold spots corresponding to the given scale level r. If one magnifies the picture near a hot spot, another more refined picture of hot and cold spots emerges. The distribution of hot and cold spots on the new scale sometimes resembles the distribution of hot and cold spots on the old one (if not precisely, then in an asymptotic way). One can speculate that this is due to a hidden group of scale symmetries admitted by the system.

To characterize the asymptotic scaling behavior of the distribution of hot and cold spots, one can define the *scaling exponents* α_i by $p_i = r^{\alpha_i}$. In [HJKPS], the authors suggested using the limit distributions of numbers α_i as $r\to\infty$ as a qualitative characteristic of the distribution of hot and cold spots. We now attempt to lay the proper mathematical foundation to make this idea rigorous.

Given a point $x\in X$ and a Borel probability measure ν on X, we define the *upper* and *lower pointwise dimensions* of ν at $x\in Z$,

$$\overline{d}_\nu(x)=\limsup_{r\to 0}\frac{\log\ \nu(B(x,r))}{\log r}\quad\text{and}\quad \underline{d}_\nu(x)=\liminf_{r\to 0}\frac{\log\ \nu(B(x,r))}{\log r}.$$

If $\underline{d}_\nu(x)=\overline{d}_\nu(x)$ we call the common value the *pointwise dimension* at x and denote it by $d_\nu(x)$. We call ν *exact dimensional* if $\overline{d}_\nu(x)=\underline{d}_\nu(x)=d_\nu(x)=d$ for ν—almost every x where d is a non-negative constant. In general one does not expect the pointwise dimension of ν to exist at a typical point even for good measures that are invariant under dynamical systems [LM, PW1]. For some dynamical systems, even when the pointwise dimension of ν does exist it is not necessarily constant and hence ν is not necessarily exact dimensional [C, PW1]. Nevertheless, *good* measures that are invariant under smooth dynamical systems with hyperbolic behavior often turn out to be exact dimensional (see Sec. V).

The *multifractal analysis* of X with respect to ν is a description of the fine-scale geometry of the set X (more precisely, the subset where the measure ν is concentrated) whose constituent components are the level sets

$$K_\alpha=\{x\in X|d_\nu(x)=\alpha\},$$

for $\alpha\geqslant 0$. We obtain a natural decomposition of the set X as

$$X=\bigcup_{-\infty<\alpha<\infty}K_\alpha\cup\{x\in X|d_\nu(x)\ \text{does not exist}\}.$$

There are several fundamental questions about this decomposition such as how large is the set of values attained by $d_\nu(x)$ and how *large* is the set of points x such that $d_\nu(x)$ does not exist?

For the maps we consider in this paper (conformal repellers and Axiom A surface diffeomorphisms), there exists an open interval of values of α such that the sets K_α are dense. Thus for these maps the decomposition of the set X is quite complicated.

To analyze this decomposition one defines the *$f_\nu(\alpha)$-spectrum for dimensions* by

$$f_\nu(\alpha)=\dim_H K_\alpha,$$

where $\dim_H K_\alpha$ denotes the Hausdorff dimension of the set K_α. The dimension spectrum is the second major object in the multifractal analysis.

Since the sets K_α are dense everywhere, one cannot replace the Hausdorff dimension in the definition of $f_\nu(\alpha)$ spectrum by a box dimension, since the box dimension of a set coincides with the box dimension of the closure of the set. This would lead to a trivial spectrum of dimensions.

It is important to emphasize that for *good* dynamical systems, the union of the sets K_α need not be all of X. Shereshevsky [Sh] showed that for some C^2 Axiom A surface diffeomorphisms, the set of points for which the pointwise dimension does not exist is dense and has a positive Hausdorff dimension for any Gibbs measure ν. In [BPS] the authors show that for *most* C^2 Axiom A surface diffeomorphisms and conformal expanding maps, and *most* Gibbs measures, the set of points for which the pointwise dimension does not exist is dense and has a full Hausdorff dimension (Hausdorff dimension is equal to the Hausdorff dimension of the basic set or the repeller).

In [HJKPS] (see also [CLP]), the authors present a heuristic argument based on the analogy with statistical mechanics to show that the Rényi-spectrum for dimensions and the $f(\alpha)$-spectrum for dimensions form a Legendre transform pair. Roughly speaking they place a uniform grid of size r over the attractor and consider the partition function

$$Z(q,r)=\sum_{i=1}^{N_r}\nu(B_i^r)^q=\sum_{i=1}^{N_r}\exp(-qE_i^r),$$

where q is the *inverse temperature* and $E_i^r=-\log\nu(B_i^r)$ is the *energy* of the grid element B_i^r [the sum is taken over those grid elements B_i for which $\nu(B_i^r)>0$]. The *free energy* of ν is defined by

$$F(q)=-\lim_{r\to 0}\frac{1}{N(r)}\log Z(q,r),$$

if the limit exists. The analogy with statistical mechanics is then used to relate the Legendre transform of F to the distribution of the numbers $\nu(B_i)$, i.e., to the dimension spectrum $f_\nu(\alpha)$. This can be made rigorous using the theory of large deviations. For all of this to make sense, one must first establish that the two spectra are smooth and strictly convex on some interval. *A priori* this seems quite amazing since in general one expects the functions $f_\nu(\alpha)$ and $R_\nu(q)$ to be *only measurable* and not even continuous.

Another application of the multifractal analysis of Gibbs measures is the study of the Lyapunov spectrum (see Sec. VI). Lyapunov exponents measure the exponential rate of divergence of infinitesimally close orbits of a smooth dynamical system. These exponents are intimately related with the global stochastic behavior of the system and hence, are fundamental invariants of the system. Lyapunov exponents are intrinsically only measurable objects and that any regularity in their behavior is unexpected and can be exploited in studying ergodic properties of the dynamical system.

In [W1] the second author studied conformal repellers and found an explicit relationship between the dimension spectrum for measure of maximal entropy and the Lyapunov spectrum (the analogous spectrum for Lyapunov exponents). He showed that for *most* conformal repellers, the Lyapunov spectrum is a real analytic and strictly convex function on an open interval. It follows that the range of the Lyapunov exponent contains an open interval of values, and hence the Lyapunov exponent attains uncountably many distinct values. For each value α in his interval, the set of points whose Lyapunov exponent is α is dense in the repeller. Thus the sets on which the Lyapunov exponent attains different values are intermingled in a very complicated way. The analogous result for the positive and negative Lyapunov exponents for an Axiom A surface diffeomorphism follows from Theorem III.1 using the same idea (see Theorem VI.3).

We believe that one can effect a complete multifractal analysis for Gibbs measures on hyperbolic sets *in arbitrary dimensions*. An obvious obstacle is that action of the map on the stable and unstable manifolds is no longer conformal. There is a paucity of techniques for computing the Hausdorff dimension for nonconformal maps.

One can consider another interesting dimension spectrum associated with the Shannon–McMillan–Brieman theorem when one computes the Hausdorff dimension of the sets where local entropy of a measure ν attains a given value. The study of this spectrum in certain cases can be reduced to the study of the $f_\nu(\alpha)$ spectrum for dimensions. There are more general notions of multifractal spectra. For example, one can replace the Hausdorff dimension by the topological entropy. See the paper by Barreira, Pesin, and Schmelling in this volume [BPS].

We refer the reader to [Pe2] for a comprehensive and systematic treatment of dimension theory in dynamical systems. The text contains detailed proofs of most of the results mentioned in this paper.

II. MULTIFRACTAL ANALYSIS OF GIBBS MEASURES ON CONFORMAL REPELLERS

In this section we effect a complete multifractal analysis for Gibbs measures on conformal repellers. Examples include Markov maps of an interval, β transformations, rational maps having hyperbolic Julia sets, and conformal toral endomorphism. We prove that the functions $f_\nu(\alpha)$ and $(1-q)\mathrm{HP}_\nu(q)$ are analytic, strictly convex on an interval, and form a Legendre transform pair, provided the measure is not the measure of maximal entropy (see Theorem II.2). In particular, this implies that the set of values attained by the point-wise dimension contains an open interval (α_1,α_2). Furthermore for each $\alpha\in(\alpha_1,\alpha_2)$, we construct a Gibbs measure ν_α such that $\nu_\alpha(K_\alpha)=1$ and thus the sets K_α are dense in the repeller. Since Gibbs measures on conformal repellers are diametrically regular, we know that the HP spectrum coincides with the Rényi spectrum.

Let M be a smooth Riemannian manifold and let g: $M\to M$ be a $C^{1+\alpha}$ map. Let J be a compact subset of M such that (i) $g(J)=J$, (ii) there exists $C>0$ and $\alpha>1$ such that $\|dg_x^n u\|\geqslant C\alpha^n\|u\|$ for all $x\in J$, $u\in T_xM$, and $n\geqslant 1$ (for some Riemannian metric on M), and (iii) that g is topologically transitive on J. In this case we say that g is a smooth *expanding* map on J. If, in addition, one assumes that there exists an open neighborhood V of J (a basin) such that $J=\{x\in V|g^n x\in V \text{ for all } n\geqslant 0\}$, we call J a *repeller*. The results in this paper do not require this extra condition on J. However, we will abuse terminology and call J a repeller even if it does not possess an open basin.

We recall some facts about expanding maps. For simplicity we assume that the map g on J is topologically mixing. In [B, Ru1], Bowen and Ruelle show that for any Hölder continuous function ξ on J there exists a unique Gibbs measure $\nu=\nu_\xi$ on J. Expanding maps have Markov partitions [Ru1, PW3] consisting of partition elements called *rectangles*, $\{R_1,\ldots,R_p\}$ of (arbitrarily small) diameter δ such that

(1) each rectangle R is the closure of its interior $\mathring{R}$,
(2) $J=\cup_i R_i$,
(3) $\mathring{R}_i\cap\mathring{R}_j=\varnothing$ for $i\neq j$, and
(4) each $g(R_i)$ is a union of rectangles R_j.

A Markov partition $\mathscr{R}=\{R_1,\ldots,R_p\}$ generates a symbolic model of the repeller by a subshift of finite type (Σ_A^+,σ), where $A=(a_{ij})$ is the transfer matrix of the Markov partition, i.e., $a_{ij}=1$ if $\mathring{R}_i\cap g^{-1}(\mathring{R}_j)\neq\varnothing$ and $a_{ij}=0$ otherwise.

In [PW3], we constructed a special Markov partition for repellers. Our construction naturally extends to hyperbolic sets [W2]. The construction is geometrically natural and simpler than other constructions of which we are aware. This construction is specially adapted to a given point (or any finite collection of points) such that the partition element containing this point also contains a *large* ball centered at the point. More precisely, let $R(x)$ denote the rectangle in $\mathscr{R}$ that contains the point x.

Theorem II.1: There are positive constants C_1, C_2 and a positive integer k such that for any $0<r\leqslant r_0$ and any $x\in X$, there exists a Markov partition $\mathscr{R}_x=\{R_1,\ldots,R_p\}$ for the map g^k such that $\mathrm{diam}(R_i)\leqslant C_2 r$ for all $i=1,\ldots,M$ and $B(x,C_1 r)\subset R(x)$.

Markov partitions allow us to define a coding map $\chi:\Sigma_A^+\to J$ such that the following diagram commutes

$$\begin{array}{ccc} \Sigma_A^+ & \xrightarrow{\ \sigma\ } & \Sigma_A^+ \\ \chi\downarrow & & \downarrow\chi. \\ J & \xrightarrow{\ g\ } & J \end{array}$$

The map χ is Hölder continuous and injective on the set of points whose trajectories never hit the boundary of any element of the Markov partition.

Given a Markov partition $\mathscr{R}=\{R_1,\ldots,R_p\}$, define the *basic sets*

$$R_{i_1\ldots i_n}=R_{i_1}\cap g^{-1}(R_{i_2})\cap\cdots\cap g^{-n+1}(R_{i_n}), \tag{1}$$

where g^{-i} denotes a branch of the inverse of g^i. By the Markov property, every basic set has the property that $R_{i_1\ldots i_n}=R_{i_1}\cap g^{-n+1}(R_{i_n})$.

A smooth map g is called *conformal* if $dg_x=a(x)\mathrm{Isom}_x$, where Isom_x denotes an isometry of the tangent space T_xM. A smooth conformal map g is called an *expanding map* if $|a(x)|>1$ for all points x. The repeller J for a conformal expanding map g is called a *conformal repeller*.

In [PW3], we established the fundamental property of Gibbs measures.

Theorem II.2: Let ξ be a Hölder continuous function on a conformal repeller J. Then the Gibbs measure for ξ with respect to g is diametrically regular.

The following are several examples of conformal repellers.

(1) *Rational Maps*. Let $R:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ be a rational map of degree $\geqslant 2$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere. The map R, being holomorphic, is clearly conformal. The Julia set J of R is the closure of the set of repelling periodic points of R [recall that a periodic point p of period m is repelling if $|(R^m)'(p)|>1$]. One says that R is *hyperbolic* (or that the

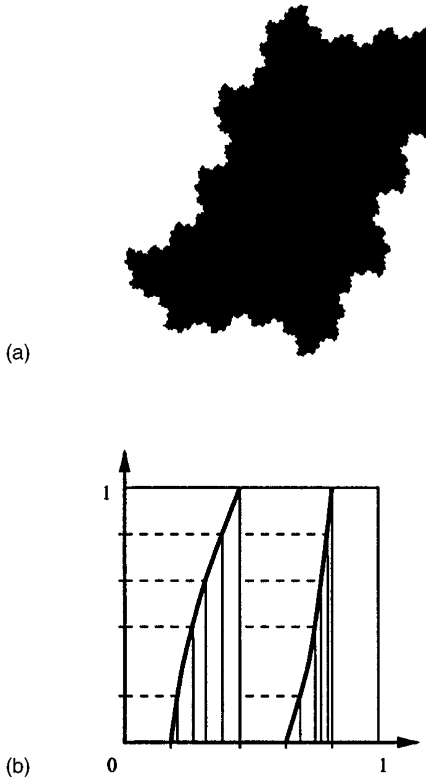

FIG. 2. (a) The boundary of the Black Spot is the Julia set for the polynomial z^2+c with $c=-\frac{1}{10}+\frac{1}{5}i$. (b) A one-dimensional Markov map.

Julia set is *hyperbolic*) if the map R is expanding on J. It is known that the map $z\mapsto z^2+c$ is hyperbolic provided $|c|<1/4$. Figure 2(a) illustrates the Julia set of a hyperbolic rational map. It is conjectured that there is a dense set of hyperbolic quadratic maps.

(2) *One-Dimensional Markov Maps*. Let $I=[0,1]$ and f be a *Markov map*. This means that there exists a finite family $I_1, I_2, \ldots I_p \subset I$ of disjoint closed intervals such that

(A) for every j, there is a subset $K=K(j)$ of indices with $f(I_j)=\cup_{k\in K} I_k \pmod 0$;

(B) for every $x\in\cup_j \mathring{I}_j$, the derivative of f exists and satisfies $|f'(x)|\geq\alpha$ for some fixed $\alpha>0$; and

(C) there exists $\lambda>1$ and $n_0>0$ such that if $f^m(x)\in\cup_j\mathring{I}_j$, for all $0\leq m\leq n_0-1$ then $|(f^{n_0})'(x)|\geq\lambda$.

Let $J=\{x\in I | f^n(x)\in\cup_j\mathring{I}_j \text{ for all } n\in\mathbb{N}\}$. The set J is a repeller for the map f. It is conformal because the domain of f is one-dimensional [see Fig. 2(b)].

(3) *Conformal Toral Endomorphism*. This is a map of a multidimensional torus defined by a diagonal matrix $(k,\ldots,k)$ where k is an integer and $|k|>1$.

(4) *Beta Transformations*. These are expanding maps of the unit interval defined by

$$x\rightarrow\beta x \pmod 1$$

for $\beta>1$. The beta transformation is intimately related to the base β expansion of real numbers.

(5) *Schottky Groups*. A Schottky group is a Kleinian group G (discrete subgroup of linear fractional transformations of $\hat{C}$) with generators $g_1,\ldots g_p$, $p\geq 1$ such that there exist $2p$ disjoint Jordan curves $\gamma_1, \gamma_1', \ldots, \gamma_p, \gamma_p'$ bounding a $2p$-connected region D for which g_j(exterior γ_j)=interior γ_j' for $j=1,\ldots,p$. The group G is free and all nontrivial elements are either hyperbolic or loxodromic (G is purely loxodromic). The factor of the region of discontinuity $\Omega(G)/G$ is a closed surface of genus p. The Koebe uniformization theorem asserts that all closed Riemann surfaces can be uniformized by Schottky groups [Kr], and thus all co-compact Fuchsian groups possess Schottky generators. One can also show that every finitely generated, free, purely loxodromic Kleinian group is Schottky [Ma].

It is easily seen that the limit set for a Schottky group G is obtained as a limit set for a Moran-like geometric construction modeled by a (one-sided) subshift of finite type [PW1;PW2]. Furthermore, the generators of G are conformal contraction maps (on their natural domains of definition) and thus the mapping induced on the limit set of G by the shift is conformal and expanding.

Ruelle [Ru2] showed that the Hausdorff dimension d of a conformal repeller J is given by *Bowen's formula*, $P(-d\log|a|)=0$, where P is the thermodynamic pressure, and that the d–Hausdorff measure is equivalent to the equilibrium measure m corresponding to $-d\log|a|$ (see Appendix B for the definition of equilibrium measure). The measure m plays a special role in the multifractal analysis and we call this measure the *measure of maximal dimension*.

We follow the statistical physics convention and identify equilibrium measures on a repeller corresponding to Hölder continuous functions with the Gibbs measures obtained by pulling them back to symbolic space (see Appendix B).

In [PW1] the authors showed that every Gibbs measure ν on a conformal repeller is exact dimensional and that for almost every $x\in J$ we have that

$$d_\nu(x)=\frac{h_\nu(g)}{\chi_\nu}, \tag{2}$$

where $h_\nu(g)$ is the Kolmogorov–Sinai entropy of g and $\chi_\nu=\int_J a(x)d\nu(x)$ is the Lyapunov exponent of ν. The set of points where (2) does not hold has zero measure but full Hausdorff dimension [BPS].

Our approach to study the pointwise dimension is to compute it with the help of the pointwise dimension of the symbolic model. The idea is to replace balls containing a point x with the basic set containing x. Let

$$\overline{\delta}_\nu(x)\equiv\limsup_{n\to\infty}\frac{\log\nu(R_{i_1\ldots i_n}(x))}{\log\operatorname{diam}(R_{i_1\ldots i_n}(x))}$$

and

$$\underline{\delta}_\nu(x)\equiv\liminf_{n\to\infty}\frac{\log\nu(R_{i_1\ldots i_n}(x))}{\log\operatorname{diam}(R_{i_1\ldots i_n}(x))}.$$

If $\overline{\delta}_\nu(x)=\underline{\delta}_\nu(x)$ we denote the common value by $\delta_\nu(x)$. We refer to this value as the *symbolic pointwise dimension* of the measure ν at x. The careful reader will observe that the symbolic pointwise dimensions are not well defined for points x

which lie on the boundary of the Markov partition. Although the boundary has measure zero with respect to any ergodic measure, it may carry a positive Hausdorff dimension. The rigorous way to define the symbolic pointwise dimensions is to define them on the symbolic model using the (pullback) measure of cylinder sets (see [PW3]).

The following result from [PW1] describes some relations between $\overline{\delta}_\nu(x)$, $\underline{\delta}_\nu(x)$ and the lower and upper pointwise dimensions at x.

Theorem II.3: Let $g:J\to J$ be a smooth conformal expanding map and let ν be any invariant probability measure (not necessarily Gibbs). Then

(1) $\overline{d}_\nu(x)\leq\overline{\delta}_\nu(x)$ for all $x\in J$; and
(2) $\underline{\delta}_\nu(x)\leq\underline{d}_\nu(x)$ for ν-almost every $x\in J$. Combining (1) and (2) yields;
(3) if $\delta_\nu(x)$ exists for ν-almost every $x\in J$, then $d_\nu(x)=\delta_\nu(x)$ for ν-almost every $x\in J$.

One can obtain a stronger version of Statement (3) in the case when ν is the Gibbs measure on J corresponding to a Hölder continuous potential φ. Let ψ be the function such that $\log\psi=\varphi-P(\varphi)$. Clearly ψ is a Hölder continuous function on J such that $P(\log\psi)=0$ and ν is the Gibbs measure for $\log\psi$. It easy follows that

$$\delta_\nu(x)=\lim_{n\to\infty}\frac{\log\nu(R_{i_1\ldots i_n}(x))}{\log\operatorname{diam}(R_{i_1\ldots i_n}(x))}=\lim_{n\to\infty}\frac{\log\Pi_{k=1}^n\psi(g^k(x))}{\log\Pi_{k=1}^n|a(g^k(x))|^{-1}},$$

in the sense that if either limit exists then the other limit exists and they coincide. In [PW3] we proved the following nontrivial theorem, which says that for Gibbs measures, the symbolic pointwise dimension coincides with the pointwise dimension of the repeller. The proof uses the fact that Gibbs measures are diametrically regular (Theorem II.2).

Theorem II.4: Let $g: J\to J$ be a conformal repeller and ν a Gibbs measure on J. Then the pointwise dimension $d_\nu(x)=c$ if and only if the symbolic pointwise dimension $\delta_\nu(x)=c$.

A. Moran cover

Using the basic sets we construct a special Moran cover $\mathfrak{U}_r$ of the repeller. Our cover is in the spirit of the cover originated by Moran in his seminal paper [Mo]. This cover has the following crucial property: Given a point $x\in J$ and a positive number r, the number of basic sets R_r^j in the Moran cover $\mathfrak{U}_r$ that have a nonempty intersection with the ball $B(x,r)$ is bounded from above by a number M, which is independent of x and r. We call this number the *Moran multiplicity factor* (see [PW1]). The Moran cover is the optimal cover in computing the Hausdorff dimension and box dimension of repellers. We make repeated use of this cover and we can not overstate its importance in our analysis.

We now construct the Moran cover. Given $0<r<1$ and a point $x\in J$, let $n(x)$ denote the unique positive integer such that

$$\prod_{k=1}^{n(x)}|a(g^k(x))|^{-1}>r,\qquad\prod_{k=1}^{n(x)+1}|a(g^k(x))|^{-1}\leq r.\tag{3}$$

It is easy to see that $n(x)\to\infty$ as $r\to 0$ uniformly in x. Fix $x\in J$ and consider the basic set $R_{i_1\ldots i_{n(x)}}\subset J$. We have $x\in R_{i_1\ldots i_{n(x)}}$, and if $x'\in R_{i_1\ldots i_{n(x)}}$ with $n(x')\geq n(x)$, then

$$R_{i_1\ldots i_{n(x')}}\subset R_{i_1\ldots i_{n(x)}}.$$

Let $R(x)$ be the largest basic set containing x with the property that $R(x)=R_{i_1\ldots i_{n(x'')}}$ for some $x''\in R(x)$ and $R_{i_1\ldots i_{n(x')}}\subset R(x)$ for any $x'\in R(x)$. The sets $R(x)$ corresponding to different $x\in J$ either coincide or are disjoint modulo their boundaries. We denote these sets by R_r^j, $j=1,\ldots,N_r$. There exist points $x_j\in J$ such that $R_r^j=R_{i_1\ldots i_{n(x_j)}}$. These sets form an *almost disjoint* cover of J (a cover where the elements of the cover have disjoint interiors) that we denote by $\mathfrak{U}_r$. If one constructs the corresponding cover on the symbolic model, one obtains a disjoint cover (see [PW3]).

Let ξ be a Hölder continuous function on J and let $\nu=\nu_\xi$ be the corresponding Gibbs measure for g. Define the one parameter family of functions φ_q, $q\in(-\infty,\infty)$ on J by $\varphi_q(x)=-T(q)\log|a(x)|+q\log\psi(x)$ where $T(q)$ is chosen such that $P(\varphi_q)=0$ [one can show that $T(q)$ exists for every $q\in\mathbb{R}$; see Lemma II.4 in the proof of Theorem II.5 below]. It is obvious that the functions φ_q are Hölder continuous.

We now state our main theorem for $C^{1+\alpha}$ conformal expanding maps. This theorem effects a complete multifractal analysis for Gibbs measures on conformal repellers.

Theorem II.5: Let $g: J\to J$ be a $C^{1+\alpha}$ conformal repeller and ν a Gibbs measure having potential $\log\psi$.

(1) The pointwise dimension $d_\nu(x)$ exists for ν-almost every $x\in J$ and

$$d_\nu(x)=\frac{\int_J\log\psi(x)d\nu(x)}{-\int_J\log|a(x)|d\nu(x)}=\frac{h_\nu(g)}{\chi_\nu},$$

where $h_\nu(g)$ is the measure theoretic entropy and $\chi_\nu=\int_J\log|a(x)|d\nu(x)$ is the Lyapunov exponent of ν (see Sec. VI).

(2) The function $T(q)$ is real analytic for all $q\in\mathbb{R}$, $T(0)=\dim_H J$, $T(1)=0$, $T'(q)\leq 0$ and $T''(q)\geq 0$ [see Fig. 3(a)].

(3) The function $\alpha(q)=-T'(q)$ attains values in the interval $[\alpha_1,\alpha_2]$, where $0\leq\alpha_1\leq\alpha_2<\infty$. The function $f_\nu(\alpha(q))=T(q)+q\alpha(q)$ [see Fig. 3(b)].

(4) If $\nu\neq m$ then the functions $f_\nu(\alpha)$ and $T(q)$ are strictly convex and form a Legendre transform pair (see Appendix B).

(5) The ν-measure of any open ball centered at points in J is positive and for any $q\in\mathbb{R}$ we have

$$T(q)=-\lim_{r\to 0}\frac{\log\inf_{\mathscr{G}_r}\Sigma_{B\in\mathscr{G}_r}\nu(B)^q}{\log r},$$

where the infimum is taken over all finite covers $\mathscr{G}_r$ of J by open balls B of radius r. For any $q>1$ (actually for any $q\neq 1$, see Remark 2) we have

$$\frac{T(q)}{1-q}=\mathrm{HP}_\nu(q)=R_\nu(q),$$

where $R_\nu(q)$ denotes the Rényi spectrum.

Sketch of Proof: Fix any $q\in\mathbb{R}$. Let ν_q denote the Gibbs measure corresponding to φ_q. Clearly, $T(0)=\dim_H J=d$. To prove Statement 1, we need the following lemma.

Lemma II.1: There exist constants $C_1>0$ and $C_2>0$ such that for all basic sets $R_{i_1\ldots i_n}$,

$$C_1\leqslant\frac{\nu_q(R_{i_1\ldots i_n})}{m(R_{i_1\ldots i_n})^{T(q)/d}\nu(R_{i_1\ldots i_n})^q}\leqslant C_2. \tag{4}$$

Proof. Since the measures ν and ν_q are Gibbs measures corresponding to the Hölder continuous functions $\log\psi$ and $-T(q)\log|a|+q\log\psi$ respectively, and m-is the Gibbs measure corresponding to function $-d\log(|a|)$, it follows from the definition of Gibbs measure [see (4) in Appendix B] that the ratios

$$\frac{\nu(R_{i_1\ldots i_n}(x))}{\Pi_{k=0}^{n-1}\psi(g^k(x))},\quad \frac{\nu_q(R_{i_1\ldots i_n}(x))}{\Pi_{k=0}^{n-1}|a((g^k(x)))|^{-T(q)}\psi(g^k(x))^q},$$

$$\frac{m(R_{i_1\ldots i_n}(x))}{\Pi_{k=0}^{n-1}|a(g^k(x))|^{-d}}$$

are bounded from below and above by constants independent of n. The lemma easily follows. ■

Given $0<r<1$ consider the Moran cover $\mathfrak{U}_r$ of the repeller J by basic sets $R_r^j=R_{i_1\ldots i_{n(x_j)}}$ with radii approximately equal to r. Let $N(x,r)$ denote the number of sets R_r^j that have a nonempty intersection with a given ball $B(x,r)$ centered at x of radius r. We have that $N(x,r)\leqslant M$, uniformly in x and r, where M is the Moran multiplicity factor.

Since the measure m is a Gibbs measure and $P(-d\log|a(x)|)=0$, there exists positive constants C_1 and C_2 such that

$$C_1\leqslant\frac{m(R_{i_1\ldots i_n}(x))}{\Pi_{k=0}^{n-1}|a(g^kx)|^{-d}}\leqslant C_2$$

(see Appendix B).

It follows from properties of the Moran cover [see (2)] that there exist positive numbers C_5 and C_6 such that for every $R_r^j\in\mathfrak{U}_r$

$$C_5r^d\leqslant m(R_r^j)\leqslant C_6r^d. \tag{5}$$

Since $\mathfrak{U}_r$ is a disjoint cover of J, we have

$$\sum_{R_r^j\in\mathfrak{U}_r}\nu_q(R_r^j)=1.$$

Summing (5) over the cover $\mathfrak{U}_r$, we obtain that there exist positive constants C_7 and C_8 such that

$$C_7\leqslant r^{T(q)}\sum_{R_r^j\in\mathfrak{U}_r}\nu(R_r^j)^q\leqslant C_8.$$

Taking logs and dividing by $\log r$, yields

$$-\lim_{r\to 0}\frac{\log\Sigma_{R_r^j\in\mathfrak{U}_r}\nu(R_r^j)^q}{\log r}=T(q). \tag{6}$$

Note that (6) holds for all $q\in\mathbb{R}$.

We now prove Statement 1 of the theorem. Given a number $\alpha\geqslant 0$, let

$$\hat{K}_\alpha=\left\{x\in J\,\middle|\,\delta_\nu(x)=\lim_{n\to\infty}\frac{\log\Pi_{k=1}^n\psi(g^k(x))}{\log\Pi_{k=1}^n|a(g^k(x))|^{-1}}=\alpha\right\}, \tag{7}$$

where $\delta_\nu(x)$ denotes the symbolic pointwise dimension. Define the *symbolic dimension spectrum*

$$\hat{f}_\nu(\alpha)=\dim_H\hat{K}_\alpha. \tag{8}$$

Given $q\in\mathbb{R}$, set

$$\alpha(q)=\frac{\int_J\log(\psi(x))d\nu_q}{\int_J\log|a(x)|^{-1}d\nu_q}.$$

We will show that this definition of $\alpha(q)$ coincides with Statement 3 in Theorem 1 (see Lemma II.4).

The following lemma allows us to compute the Hausdorff dimension of the set $\hat{K}_{\alpha(q)}$.

Lemma II.2:

(1) The measure $\nu_q(\hat{K}_{\alpha(q)})=1$.
(2) The pointwise dimension $d_{\nu_q}(x)=T(q)+q\alpha(q)$ for ν_q-almost all $x\in\hat{K}_{\alpha(q)}$ and $\overline{d}_{\nu_q}(x)\leqslant T(q)+q\alpha(q)$ for all $x\in\hat{K}_{\alpha(q)}$.
(3) The Hausdorff dimension $\dim_H\hat{K}_{\alpha(q)}=T(q)+q\alpha(q)$.

Sketch of Proof: The first and third statements are easy consequences of the second statement. To compute the pointwise dimension $d_\nu(x)$ we use the Birkhoff ergodic theorem applied to the potential $\psi_q(x)$. To obtain the upper bound for $\overline{d}_{\nu_q}(x)$ for all $x\in\hat{K}_{\alpha(q)}$ we use the Moran cover and the Gibbs property of the measure ν_q. ■

It immediately follows from (6) that $T(1)=0$ and thus $\nu=\nu_1$. The first statement of the theorem now follows from Lemma II.2 and Theorem II.2.

We now prove Statements 2, 3, and 4 of the theorem. We first observe that $\dim_H K_{\alpha(q)}=T(q)+q\alpha(q)$. Since $\nu_q(K_{\alpha(q)})=1$, this is a consequence of Lemmas II.2, II.3, and the following general result.

Lemma II.3: Let (X,ρ) be a complete separable metric space of finite topological dimension with metric ρ, and let μ be a Borel probability measure. If $Z_\beta=\{x\in X|\underline{d}_\mu(x)=\overline{d}_\mu(x)=\beta\}$ and $\mu(Z_\beta)>0$, then $\dim_H Z_\beta=\beta$.

We also require the following lemmas.

Lemma II.4:

(1) The function $T(q)$ is real analytic for all $q\in\mathbb{R}$.

(2) The function $T(q)$ is convex. It is not strictly convex if and only if $\nu=m$.

(3) For all q we have $\alpha(q)=-T'(q)$.

Proof.

The proof of (1) uses the real analyticity of pressure and the inverse function theorem. The proof of (2) uses Ruelle's second derivative formula for pressure [Ru1]. To prove (3) let $\varphi_{q,r}(x)=-r\log|a(x)|+q\log\psi(x)$ and recall that $\varphi_q(x)=-T(q)\log|a(x)|+q\log\psi(x)$. Since $P(\varphi_q)=0$ for all q we have

$$\frac{d}{dq}P(\varphi_q)=\frac{\partial P(\varphi_{q,r})}{\partial q}+\left.\frac{\partial P(\varphi_{q,r})}{\partial r}\right|_{T(q)}T'(q)=0.$$

Using the well known formula for the derivative of pressure (see Appendix B) we obtain that

$$T'(q)=-\frac{\left.\frac{\partial P(\varphi_{q,r})}{\partial r}\right|_{r=T(q)}}{\left.\frac{\partial P(\varphi_{q,r})}{\partial q}\right|_{r=T(q)}}=-\frac{\int_J\log(\psi(x))d\nu_q}{\int_J\log|a(x)|d\nu_q}$$

$$=-\alpha(q).$$

■

It follows from Lemma II.4 that the function $\alpha(q)$ is analytic and $\alpha'(q)=-T''(q)<0$. Hence, the range of the function $\alpha(q)$ contains an interval. This implies Statements 2, 3, and 4.

The proof of the Statement 5 of the theorem uses several covering arguments involving the Moran cover and some refinements. The proof also uses (6) and the fact that Gibbs measures are diametrically regular.

B. Remarks

(1) It follows from our proof that for every $\alpha_1\leqslant\alpha(q)\leqslant\alpha_2$ there is a unique Gibbs measure ν_q on J such that $\nu_q(K_{\alpha(q)})=1$ and $d_{\nu_q}(x)=f_\nu(\alpha(q))$ for every point $x\in K_{\alpha(q)}$.

(2) For an arbitrary Borel probability measure ν on a metric space X, the HP spectrum is not *a priori* defined for $q<1$. One problem is that the measure of some small balls may be zero. However, if all balls have positive measure (as in the case of Gibbs measures for conformal repellers), the definition of the HP spectrum for all $q\neq 1$ makes formal sense although the integral may be infinite. In our proof of Statement (5) in Theorem II.5, we actually show that for all $q\neq 1$ (not just for $q>1$ as stated), the function $T(q)/(1-q)$ coincides with this extended definition of $\mathrm{HP}_\nu(q)$. In particular this implies that $\mathrm{HP}_\nu(q)$ is well defined for all $q\neq 1$.
The case $q=1$ is treated in Remark (3).

(3) We define the notion of *information dimension*. Let ξ be a finite partition of the space X. Given a Borel finite measure ν on X, the entropy of ξ with respect to ν is defined as

$$H_\nu(\xi)\overset{\text{def}}{\equiv}-\sum\nu(C_\xi)\log\nu(C_\xi),$$

where C_ξ is an element of the partition ξ. Given a positive number ε, we set

$$H_\nu(\varepsilon)=\inf_\xi\{H_\nu(\xi):\text{diam }\xi\leqslant\varepsilon\},$$

where diam ξ=max diam C_ξ.

We define the *information dimension* of ν by

$$I(\nu)\overset{\text{def}}{\equiv}\lim_{\varepsilon\to 0}\frac{H_\nu(\varepsilon)}{\log 1/\varepsilon}$$

(provided that the limit exists).

In [Y], Young showed that if $\underline{d}_\nu(x)=\overline{d}_\nu(x)=d$ for ν-almost every $x\in X$ then $I(\nu)=d$, and hence is equal to the Hausdorff dimension of ν.

Assume that the measure ν is diametrically regular. It is believed that in *good* cases

$$I(\nu)=\lim_{q\to 1+}R_\nu(q)=\lim_{q\to 1+}\mathrm{HP}_\nu(q).$$

Since the function $T(q)$ is differentiable the limit

$$\lim_{q\to 1}\frac{T(q)}{1-q}$$

exists and is equal to $-T'(1)=\alpha(1)$. It follows from Statement 5 of Theorem 1 that

$$-T'(1)=\lim_{r\to 0}\frac{\log\inf_{\mathscr{G}_r}\Sigma_{B\in\mathscr{G}_r}\nu(B)\log\nu(B)}{\log r},$$

where the infinum is taken over all finite covers $\mathscr{G}_r$ of J by open balls of radius r. This implies that

$$f_\nu(\alpha(1))=\alpha(1)=-T'(1)=I(\nu).$$

III. MULTIFRACTAL ANALYSIS OF GIBBS MEASURES ON BASIC SETS OF AXIOM A DIFFEOMORPHISMS

In this section we effect a complete multifractal analysis for Gibbs measures on basic sets Λ of a $C^{1+\alpha}$ Axiom A surface diffeomorphisms. We follow the approach suggested by the authors in [PW3].

A. Review of hyperbolic dynamics

Let M be a smooth surface and $f:M\to M$ a $C^{1+\alpha}$ diffeomorphism (i.e., f is a $C^{1+\alpha}$ invertible map whose inverse is of class $C^{1+\alpha}$). A compact f-invariant subset $\Lambda\subset M$ is called *hyperbolic* if there exists a continuous splitting of the tangent bundle $T_\Lambda M=E^s\oplus E^u$ into two one-dimensional subspaces and constants $C>0$ and $0<\lambda<1$ such that for every $x\in\Lambda$

(1) $dfE^s(x)=E^s(f(x))$, $dfE^u(x)=E^u(f(x))$;
(2) for all $n\geqslant 0$

$$\|df^n v\|\leqslant C\lambda^n\|v\|\quad\text{if } v\in E^s(x),$$

$$\|df^{-n}v\|\leqslant C\lambda^n\|v\|\quad\text{if } v\in E^u(x).$$

The subspaces $E^s(x)$ and $E^u(x)$ are called *stable* and *unstable subspaces* at x, respectively. Define the continuous functions $a^s(x)=-\|df|E^s(x)\|$ and $a^u(x)=\|df|E^u(x)\|$.

It is well known (see for example, [KH]) that for every $x\in\Lambda$ one can construct one-dimensional local *stable* and *unstable local manifolds*, $W^s_{\rm loc}(x)$ and $W^u_{\rm loc}(x)$ that have the following properties:

(3) $x\in W^s_{\rm loc}(x), x\in W^u_{\rm loc}(x)$;

(4) $T_xW^s_{\rm loc}(x)=E^s(x), T_xW^u_{\rm loc}(x)=E^u(x)$;

(5) $f(W^s_{\rm loc}(x))\subset W^s_{\rm loc}(f(x)), f^{-1}(W^u_{\rm loc}(x)) \subset W^u_{\rm loc}(f^{-1}(x))$;

(6) there exist $K>0$ and $0<\kappa<1$ such that for every $n\geqslant 0$

$$\rho(f^n(y),f^n(x))\leqslant K\kappa^n\rho(y,x) \quad \text{for all} \quad y\in W^s_{\rm loc}(x),$$

and

$$\rho(f^{-n}(y),f^{-n}(x))\leqslant K\kappa^n\rho(y,x) \quad \text{for all} \quad y\in W^u_{\rm loc}(x),$$

where ρ is the distance in M induced by the Riemannian metric.

A hyperbolic set Λ is called *locally maximal* if there exists a neighborhood U of Λ such that for any closed f-invariant subset $\Lambda'\subset U$ we have $\Lambda'\subset\Lambda$. In this case

$$\Lambda=\bigcap_{-\infty<n<\infty} f^n(U).$$

A point $x\in M$ is called *nonwandering* if for each neighborhood U of x there exists $n\geqslant 1$ such that $f^n(U)\cap U\neq\emptyset$. We denote by $\Omega(f)$ the set of all nonwandering points of f. It is a closed f-invariant set. A diffeomorphism f is called an *Axiom A* diffeomorphism if $\Omega(f)$ is a locally maximal hyperbolic set. If f is an *Axiom A* diffeomorphism then $\Omega(f)$ can be decomposed into a finite number of disjoint closed f-invariant sets, $\Omega(f)=\Lambda_1\cup\ldots\cup\Lambda_n$, such that $f|\Lambda_i$ is topologically transitive. Each set Λ_i is said to be a *basic set* of f. See [KH] for a more complete description.

Let ξ be a Hölder continuous function on Λ and let $\nu=\nu_\xi$ be the Gibbs measure for f corresponding to ξ. We remind the reader that a finite cover $\mathscr{R}=\{R_1,\ldots,R_p\}$ of Λ is called a *Markov partition* for f if

(1) $\mathring{R}_i\cap\mathring{R}_j=0$ unless $i=j$;

(2) for each $x\in\mathring{R}_i\cap f^{-1}(\mathring{R}_i)$ we have

$$f(W^s_{\rm loc}(x)\cap R_i)\subset W^s_{\rm loc}(f(x))\cap R_j,$$

$$f(W^u_{\rm loc}(x)\cap R_i)\supset W^u_{\rm loc}(f(x))\cap R_j.$$

It is well known that Gibbs measures on hyperbolic sets have a local product structure [Ru1]. We state this fact in the following proposition. Let $\mathscr{R}$ be a Markov partition of Λ with transition matrix $A=(a_{i,j})$. Denote by Σ_A the set of all allowable two-sided sequences of integers $(\ldots i_{-2}i_{-1}i_0i_1\ldots)$, i.e., $a_{i_ni_{n+1}}=1$ for every n. We define the coding map χ: $\Sigma_A\to\Lambda$ by

$$\chi(\omega)=x=\bigcap_{n=-\infty}^{\infty} R_{i_{-n}\ldots i_n}.$$

Let ν be the Gibbs measure corresponding to a Holder continuous function ϕ on Λ and μ the pullback of ν to Σ_A.

Proposition III.1: There exist positive constants K_1, K_2, and r_0 with the following properties: such that for any point $x\in\Lambda$, there exist measures ν^s_x and ν^u_x on $W^s_{\rm loc}(x)\cap R(x)$ and $W^u_{\rm loc}(x)\cap R(x)$ respectively such that for any Borel sets $E\subset W^s_{\rm loc}(x)\cap R(x)$ and $F\subset W^u_{\rm loc}(x)\cap R(x)$ we have that

$$K_1(\nu^s_x(E)\times\nu^u_x(F))\leqslant\nu(E\times F)\leqslant K_2(\nu^s_x(E)\times\nu^u_x(F)).$$

In other words, locally, the measure ν near a point $x\in\Lambda$ is equivalent to the direct product of measures ν^s_x and ν^u_x. The proof relies on the fact that every rectangle $R(x)$ is *essentially* a direct product. We now give a brief description of how to construct the measures ν^s_x and ν^u_x. One can show that these measures are diametrically regular for every x. This implies that r is also diametrically regular.

Define Σ^-_A to be the set of all allowable one-sided sequences of integers $(\ldots i_{-2}i_{-1}i_0)$, i.e., $a_{i_{n+1}i_n}=1$ for every $n\geqslant 0$. Similarly define Σ^+_A to be the set of all allowable one-sided sequences of integers $(i_0i_1i_2\ldots)$, i.e., $a_{i_ni_{n+1}}=1$ for every $n\geqslant 0$. We note that the coding of every point $y\in W^s_{\rm loc}(x)\cap R(x)$ begins with the same integer i_0.

We define the Hölder continuous function ψ^s on Σ^-_A by

$$\log\psi^s(\omega)=-\lim_{n\to\infty}\log\frac{\mu(C_{i_{-n}\ldots i_{-1}})}{\mu(C_{i_{-n}\ldots i_0})},$$

where $\omega=(\ldots i_{-1}i_0)$. One can show that the measure ν^s_x is the pushforward of the restriction to C_{i_0} of the Gibbs measure on Σ^-_A for $\log\psi^s$. Similarly one can show that the measure ν^u_x is the pushforward of the restriction to C_{i_0} of the Gibbs measure on Σ^+_A for the function $\log\psi^u$ defined by

$$\log\psi^u(\omega)=-\lim_{n\to\infty}\log\frac{\mu(C_{i_1\ldots i_n})}{\mu(C_{i_0\ldots i_n})},$$

where $\omega=(i_0i_1\cdots)$. We have that $P(\log\psi^s)=0$ and $P(\log\psi^u)=0$.

Let $a^s(x)$ and $a^u(x)$ be the contraction and expansion coefficients of f along the stable and unstable directions, and t^s and t^u the unique roots of Bowen's equations $P(t\log|a^s(x)|)=0$ and $P(-t\log|a^u(x)|)=0$). Using the above codings, we pull back the functions $t^s\log|a^s(x)|$ and $-t^u\log|a^u(x))|$ to Σ^-_A and Σ^+_A respectively. Let μ^s and μ^u be the Gibbs measures corresponding to these functions and let m^s_x and m^u_x be the push forward of the restriction to C_{i_0} of these measures. The measures m^s_x and m^u_x live on $W^s_{\rm loc}(x)\cap R(x)$ and $W^u_{\rm loc}(x)\cap R(x)$ respectively.

We define the one parameter family of functions φ^s_q, $q\in(-\infty,\infty)$ on Σ^-_A by

$$\varphi^s_q=T^s(q)\log|a^s\circ\chi|+q\log\psi^s,$$

where $T^s(q)$ is chosen such that $P(\varphi^s_q)=0$. Similarly, we define the one parameter family of functions φ^u_q, $q\in(-\infty,\infty)$ on Σ^+_A by

$$\varphi^u_q=-T^u(q)\log|a^u\circ\chi|+q\log\psi^u,$$

where $T^u(q)$ is chosen such that $P(\varphi^u_q)=0$. We set

$$T(q)=T^s(q)+T^u(q).$$

Consider the Gibbs measures μ^s_q on Σ^-_A and μ^u_q on Σ^+_A corresponding to functions φ^s_q and φ^u_q and let ν^s_q and ν^u_q be their push forwards to $W^s_{\rm loc}(x)\cap R(x)$ and $W^u_{\rm loc}(x)\cap R(x)$, respectively. Let $\nu_q=\nu^s_q\times\nu^u_q$ be the product measure on $R(x)$.

We wish to identify every point in Λ with its symbolic representative. This cannot be done on the boundary of the Markov partition. However, for the sake of clarity of our statements we will assume that this identification holds everywhere (see [Pe2] for precise statements). We now state our main result that establishes the multifractal analysis for Gibbs measures (corresponding to Hölder continuous functions) for basic sets of Axiom A surface diffeomorphisms.

Given $\alpha\geqslant 0$, consider the set

$$K_\alpha=\{x\in X\,|\,d_\nu(x)=\alpha\}$$

and then the $f_\nu(\alpha)$-spectrum for dimensions

$$f_\nu(\alpha)=\dim_H K_\alpha .$$

Theorem III.1: [PW3]

(1) The pointwise dimension $d_\nu(x)$ exists for ν-almost every $x\in\Lambda$ and

$$d_\nu(x)=\frac{\int_{\Sigma_A}\log\psi(x)d\mu(x)}{\int_\Lambda \log|a^u(x)|d\nu(x)}-\frac{\int_{\Sigma_A}\log\psi(x)d\mu(x)}{\int_\Lambda \log|a^s(x)|d\nu(x)}$$
$$=h_\nu(f)\left(\frac{1}{\lambda^+_\nu}-\frac{1}{\lambda^-_\nu}\right),$$

where $h_\nu(f)$ is the measure theoretic entropy of f and λ^+_ν, λ^-_ν are positive and negative values of the Lyapunov exponent of ν (see the definition of the Lyapunov exponent in Section VI).

The function $T(q)$ is real analytic for all $q\in\mathbb{R}$, $T(0)=\dim_H\Lambda$, $T(1)=0$, $T'(q)\leqslant 0$, and $T''(q)\geqslant 0$ [see Fig. 3(a)].

The function $\alpha(q)=-T'(q)$ attains all values in an interval $[\alpha_1,\alpha_2]$ where $0\leqslant\alpha_1\leqslant\alpha_2<\infty$. The function $f_\nu(\alpha(q))=T(q)+q\alpha(q)$ [see Fig. 3(b)].

If the measure $\nu|R(x)$ is not equivalent to the measure $m^s_x\times m^u_x$, then the functions $f_\nu(\alpha)$ and $T(q)$ are strictly convex and form a Legendre transform pair.

The ν measure of any open ball centered at points in Λ is positive, and for any $q\in\mathbb{R}$ we have

$$T(q)=-\lim_{r\to 0}\frac{\log\inf_{\mathscr{G}_r}\Sigma_{B\in\mathscr{G}_r}\nu(B)^q}{\log r}$$

where the infimum is taken over all finite covers $\mathscr{G}_r$ of Λ by open balls of radius r. For every $q>1$ (actually for any $q\neq 1$, see Remark 3)

$$\frac{T(q)}{1-q}={\rm HP}_q(\nu)=R_q(\nu).$$

Sketch of Proof. Applying a theorem of Manning and McCluskey [MM] (a result like this is only known in dimension two) we have that

$$T(0)=T^s(0)+T^u(0)=t^s+t^u=\dim_H\Lambda.$$

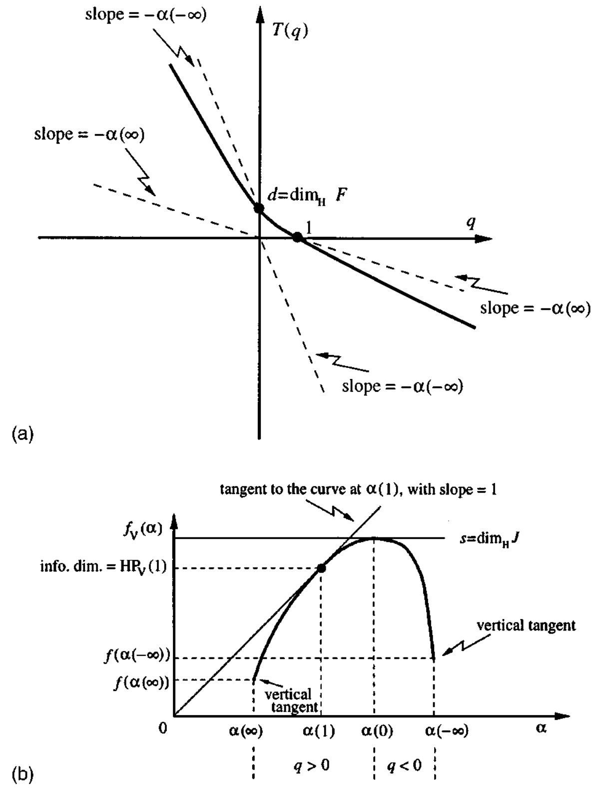

FIG. 3. (a) Graph of $T(q)$. (b) Graph of $f_\nu(\alpha)$.

We denote by $\mathfrak{U}^-_r=\{C^{(j,-)}_r\}$ and $\mathfrak{U}^+_r=\{C^{(j,+)}_r\}$ the Moran covers of Σ^-_A and Σ^+_A respectively (see Sec. II). Repeating arguments in the proof of Theorem II.5, we obtain that [see (5) in Sec. II]

$$T^s(q)=-\lim_{r\to 0}\frac{\log\Sigma_{C^{(j,-)}_r\in\mathfrak{U}^-_r}\mu^s(C^{(j,-)}_r)^q}{\log r}\tag{9}$$

and

$$T^u(q)=-\lim_{r\to 0}\frac{\log\Sigma_{C^{(j,+)}_r\in\mathfrak{U}^+_r}\mu^u(C^{(j,+)}_r)^q}{\log r}.$$

In particular, $T(1)=T^s(1)+T^u(1)=0$. We first sketch the proof of Statement 1 of the theorem.

Given a number $\alpha\geqslant 0$, let

$$\hat{K}_\alpha=\left\{x\in\Lambda\,\middle|\,\lim_{n\to\infty}\left(\frac{\Sigma^0_{k=-n}\log\psi^s(f^kx)}{\Sigma^0_{k=-n}\log|a^s(f^kx)|}-\frac{\Sigma^n_{k=0}\log\psi^u(f^kx)}{\Sigma^n_{k=0}\log|a^u(f^kx)|}\right)=\alpha\right\}.\tag{10}$$

Define the *symbolic dimension spectrum*

$$\hat{f}_\nu(\alpha)=\dim_H\hat{K}_\alpha .$$

Given $q\in\mathbb{R}$, set

$$\alpha^s(q) = \frac{\int_{\Sigma_A^-} \log(\psi^s(x))\, d\mu_q^s}{\int_\Lambda \log|a^s(x)|\, d\nu_q^s},$$

$$\alpha^u(q) = -\frac{\int_{\Sigma_A^+} \log(\psi^u(x))\, d\mu_q^u}{\int_\Lambda \log|a^u(x)|\, d\nu_q^u},$$

$$\alpha(q) = \alpha^s(q) + \alpha^u(q).$$

It is not hard to show that

$$\alpha^s(q) = \alpha^u(q) = h_{\mu_q}(f).$$

We have the following lemma that allows us to compute the Hausdorff dimension of the set $\hat{K}_\alpha(q)$.

Lemma III.1: For every $q \in \mathbb{R}$, we have

(1) the measure $\nu_q(\hat{K}_{\alpha(q)} \cap R(x)) = 1$;
(2) the pointwise dimension $d_{\nu_q}(y) = T(q) + q\alpha(q)$ for ν_q-almost all $y \in \hat{K}_{\alpha(q)} \cap R(x)$ and the upper pointwise dimension $\overline{d}_{\nu_q}(y) \leqslant T(q) + q\alpha(q)$ for all $y \in \hat{K}_{\alpha(q)} \cap R(x)$; and
(3) the Hausdorff dimension $\dim_H \hat{K}_{\alpha(q)} \cap R(x) = T(q) + q\alpha(q)$.

The proofs of these statements are very similar to the proofs of the analogous statements in Lemma II.2.

We also need the following key theorem, which is the analog of Theorem II.4.

Theorem III.2:

(1) For every $q \in \mathbb{R}$ and every $x \in \hat{K}_{\alpha(q)}$ we have that $d_\nu(x) = \alpha(q)$.
(2) If $d_\nu(x) = \alpha(q)$ then $x \in \hat{K}_{\alpha(q)}$.

Sketch of Proof: Applying arguments in the proof of Theorem II.4 to the measures ν^s and ν^u and using the facts that the measure ν is locally equivalent to the direct product of measures ν^s and ν^u (Proposition III.1), one obtains that the limit

$$\lim_{r \to 0} \frac{\log\ \nu(B(x,r))}{\log r}$$

exists if and only if the limit in (10) exists and they attain the same value. ■

The above arguments imply that the function $\alpha(q) \geqslant 0$ for all q. Since $T(1) = 0$, we have that $\nu|R(x) = \nu_1$. The first statement of the theorem now follows from Lemma III.1, Theorem III.2, and the observation that

$$\alpha^s(1) = \frac{h_\nu(f)}{\lambda_\nu^-} \quad \text{and} \quad \alpha^u(1) = \frac{h_\nu(f)}{\lambda_\nu^+}.$$

Theorem III.2 implies that $K_{\alpha(q)} = \hat{K}_{\alpha(q)}$. Hence, $f_\nu(\alpha(q)) = \dim_H\ K_{\alpha(q)} = T(q) + q\alpha(q)$. Applying Lemma II.4 to the pairs of functions $(\alpha^s(q), T^s(q))$ and $(\alpha^u(q), T^u(q))$, we obtain that the functions $\alpha(q)$ and $T(q)$ also satisfy the conclusions of Lemma II.4. This implies Statements 2, 3, and 4. Since the measure ν is diametrically regular and has local product structure, one can use (9) to prove Statement 5 by repeating arguments in the proof of the Statement 5 of Theorem II.5. ■

B. Remarks

(1) For any $q \in \mathbb{R}$ and $x \in \Lambda$ there exists a measure ν_q on $R(x)$ such that $\nu_q(K_{\alpha(q)} \cap R(x)) = 1$ and $d_{\nu(q)}(x) = T(q) + q\alpha(q)$ for $\nu(q)$-almost every $x \in K_{\alpha(q)} \cap R(x)$.
(2) Assume that the measure $\nu|R(x)$ is equivalent to $m_x^s \times m_x^u$ for any $x \in \Lambda$. One can easily show that $T(q) = (1-q)\dim_H \Lambda$ (thus $T(q)$ is a linear function). This implies that $f_\nu(\dim_H \Lambda) = \dim_H \Lambda$ and $f_\nu(\alpha) = 0$ for all $\alpha \neq \dim_H \Lambda$.
(3) As in the case of conformal repellers [see Remark (2) after the proof of Theorem II.5], we note that Statement (5) of Theorem III.1 allows us to extend the notion of the HP spectrum and Rényi spectrum for dimensions for any $q \neq 1$ (not just for $q > 1$ as stated). We actually show that for all $q \neq 1$, the function $T(q)/(1-q)$ coincides with this extended definition of $\mathrm{HP}_\nu(q)$. In particular this implies that $\mathrm{HP}_\nu(q)$ is well defined for all $q \neq 1$.
(4) The case $q = 1$ is treated in Remark (4).
(5) As in the case of conformal repellers [see Remark (3) after the proof of Theorem II.5] one can show that

$$f_\nu(\alpha(1)) = \alpha(1) = -T'(1) = I(\nu),$$

where $I(\nu)$ is the information dimensions of ν [see Remark (3) in II]. In particular, $I(\nu) = \dim_H \nu$.

IV. LARGE DEVIATIONS AND ALTERNATIVE APPROACHES TO THE MULTIFRACTAL ANALYSIS

Our proof of Theorems II.2 and III.1 did not use any results in the theory of large deviations. This is in contrast to most multifractal analyses in the literature (including [CLP, Si, Lo]) which make essential use of results in the theory of large deviations.

However, by combining our smoothness and convexity results for $T(q)$ in Theorems II.2 and III.1 with (6) and (9), we have verified all the hypotheses needed to apply a large deviation theorem of Ellis [E] and obtain an interesting formula for the dimension spectrum.

More precisely, consider the family of random variable $X_j^r = \log\ \nu(R_r^j)$, where j has been picked uniformly from $1, \ldots, N_r$. The moment generating function of X_n is $c_r(q) = \mathbb{E}(\exp(qX_n)) = (1/N_r)\Sigma_{B_j^r \in \mathfrak{U}_r} \nu(B_j^r)^q$, where $\mathbb{E}(X)$ denotes the expected value of the random variable X. Therefore, (6) implies that

$$\lim_{r \to 0} \frac{\log c_r(q)}{\log r} = T(q) - T(0),$$

which by Theorem II.5 is smooth and convex. Thus, the assumptions of Theorem II.2 in [E] are met with $a_n = \log 1/r$. Recall from Theorem II.5 that the Legendre transform of $T(q)$ is the (dimension spectrum) function $f_\nu(\alpha)$. The following theorem is a corollary of Ellis' theorem (see [Ri]), and gives a *counting* approach to the multifractal analysis.

Theorem IV.1: Let ν be the Gibbs measure on J corresponding to a Hölder continuous function ξ. If $\nu \neq m$, then

$$f_\nu(\alpha)=\lim_{\varepsilon\to 0}\lim_{r\to 0}\frac{\log N_r(\alpha,\varepsilon)}{\log 1/r},$$

where $N_r(\alpha,\varepsilon)$ is the number of sets $R_r^j \in \mathfrak{U}_r$ such that $\alpha-\varepsilon<\nu(R_r^j)\leqslant\alpha+\varepsilon$.

There are clearly close connections between the theory of large deviations, classical statistical physics, and our approach. Recall the heuristic argument of [HJKPS] based on ideas in classical statistical physics justifying the multifractal analysis (see Sec. I). The normalized partition function $Z(q,r)/N_r$ corresponds to the moment generating function $c_r(q)$ for the uniformly chosen random variables $\log \nu(B_j^r)$, where the *energy* E_i of the element $\nu(B_i^r)$ is $-\log \nu(B_i^r)$. The free energy $F(q)$ of ν coincides with $\lim_{r\to 0}\log c_r(q)/\log r$, which in our language is $(1-q)(T(q)-T(0))=(1-q)(R_\nu(q)-R_\nu(0))$. The Legendre transform of $c(q)$ is the level one entropy function that provides refined information on the convergence of the quotients $\log \nu(B_i^r)/\log r$. In the language of statistical physics, this is the Legendre transform of the free energy. In our language the Legendre transform of $T(q)$ is the dimension spectrum $f_\nu(\alpha)$.

Simpelaere [Si] first effected the multifractal analysis of Gibbs measures for Axiom A surface diffeomorphisms. We now outline his approach. His proof exploits the fact that the restriction of the diffeomorphism to the stable and unstable leaves are one-dimensional *expanding-like* maps and that a Gibbs measure ν locally has a product structure, i.e., locally $\nu\approx\nu^s\times\nu^u$.

Simpelaere first shows that the local product structure of the hyperbolic set implies that the free energy is additive with respect to the stable and unstable splitting, i.e., $F(q)=F^u(q)+F^s(q)$, where $F^u(q)$ and $F^s(q)$ are the unstable and stable free energies defined using the measures ν^u and ν^s respectively.

His argument strongly depends on a special family of grids which is adapted to the hyperbolic splitting. *A priori*, it is not at all clear whether the free energy is additive for other families of grids. For example, if a hyperbolic set is embedded in the plane (e.g., the Plykin attractor), he does not discuss whether the free energy additive with respect to the standard (x,y) grid.

He then shows that $F^u(q)$ and $F^s(q)$ satisfy the following variational principles

$$F^u(q)=\inf_{\rho\in M(J,g)}\left(\frac{h_\rho(g)-q\int_J\psi^u d\rho}{\int_J a(x)d\rho}\right)$$

and

$$F^s(q)=\inf_{\rho\in M(J,g)}\left(\frac{h_\rho(g)-q\int_J\psi^s d\rho}{\int_J a(x)d\rho}\right),$$

where $M(J,g)$ denotes the space of g-invariant Borel probability measures on J and ψ^u, ψ^s denote the projections of ψ onto the unstable and stable directions, respectively. It follows that $P(q\psi^u-F^u(q)a)=0$ and $P(q\psi^s-F^s(q)a)=0$. Let ν_q^u and ν_q^s denote the Gibbs measures for the potentials $q\psi^u-F^u(q)a$ and $q\psi^s-F^s(q)a$, respectively. Using techniques in symbolic dynamics and thermodynamic formalism (i.e., the smoothness of pressure, derivative of pressure), he shows that F^u and F^s (and thus F) are real analytic and convex.

For $\alpha=F'(q)$, write $\alpha=\alpha^u+\alpha^s$, where $\alpha^u=(F^u)'(q)$ and $\alpha^s=(F^s)'(q)$. If $g(\alpha)$, $g^u(\alpha)$ and $g^s(\alpha)$ denote the Legendre transform of F, F^u and F^s respectively, then a formal property of the Legendre transform implies that $g(\alpha)=g^u(\alpha)+g^s(\alpha)$. He now needs to identify $g(\alpha)$ with the dimension spectrum $f_\nu(\alpha)$.

Applying a construction of measures, as in [CLP], he establishes (using the mass distribution principle) the lower estimate $\dim_H(K_\alpha)\geqslant f_\nu(\alpha)$. A crucial step of Simpelaere's approach is to apply Ellis' large deviation theorem to obtain the upper estimate. Thus, he obtains that $\dim_H(K_\alpha)=f_\nu(\alpha)$. Finally, *a posteriori*, he shows that $g^u(\alpha^u)=\dim_H(\nu_q^u)$ and $g^s(\alpha^s)=\dim_H(\nu_q^s)$, and thus $g(\alpha)=f_\nu(\alpha)=\dim_H(\nu_q^u)+\dim_H(\nu_1^s)=\dim_H(\nu_q^u\times\nu_q^s)\equiv\dim_H(\nu_q)$. This last fact exploits the smoothness of the stable and unstable foliations in the two-dimensional case (which in general is false in higher dimensions).

The theorem of Ellis allows one to obtain refined information on the distribution of $\log \nu(B_n^r)/\log r$ as $r\to 0$. The large deviation theorem gives the following estimates of the cardinality of *good* covers of the set K_α, which allows one to estimate the Hausdorff dimension of the sets K_α using covers and to identify the Legendre transform of the free energy $I(z)$ with the $f_\nu(\alpha)$ spectrum:

$$\limsup_{r\to 0}\frac{\log\left(\#n:1\leqslant n\leqslant N_r \text{ such that } \dfrac{\log\ \nu(B_n^r)}{\log r}\in[a,b]/N_r\right)}{\log r}\leqslant -I[a,b]$$

and

$$\liminf_{r\to 0}\frac{\log\left(\#n:1\leqslant n\leqslant N_r \text{ such that } \dfrac{\log\ \nu(B_n^r)}{\log r}\in(a,b)/N_r\right)}{\log r}\geqslant -I(a,b),$$

where $I(A)=\inf\{I(z)|z\in A\}$ and $I(z)$ is the Legendre transform of his free energy $F(q)$.

Like Simpelaere's approach, our method also exploits the fact that the restriction of the diffeomorphism to the stable and unstable leaves are one-dimensional *expanding-like* maps and that a Gibbs measure ν locally has a product structure. We first effect the multifractal analysis for ν^u and ν^s separately and then carefully combine the two pieces to obtain the multifractal analysis of ν.

An important technical tool in our analysis is the fact that Gibbs measures on conformal repellers and Axiom A surface diffeomorphisms are diametrically regular. This allows us to prove the remarkable fact (which has not been observed by other authors) that for Gibbs measures, the pointwise dimension of the measure ν at a point $x\in\Lambda$ exists *if and only if* the symbolic pointwise dimension of the pullback measure on the symbolic space Σ_A exists at the point ω [where $\chi(\omega)=x$], and the values coincide. Surprisingly, this means that the symbolic model carries all of the essential information needed to compute the pointwise dimensions of the hyperbolic set.

Our crucial result that Gibbs measures are diametrically regular allows us to work directly with the HP spectrum (defined using the free energy of covers) and also shows that it coincides with the Rényi spectrum (defined using the free energy of partitions). Furthermore, we prove that the free energy of partitions (grids) and covers is additive, regardless of which grid or cover is used.

V. THE ECKMANN-RUELLE CONJECTURE AND "COUNTEREXAMPLES"

Eckmann and Ruelle conjectured that an ergodic measure λ which is invariant under a $C^{1+\alpha}$ diffeomorphism with nonzero Lyapunov exponents is exact dimensional (and hence, $d_\lambda(x)=$const almost everywhere). This has been one of the most challenging problems in the interface between dimension theory and dynamical systems.

In [Y], Young obtained the positive solution for the Eckmann–Ruelle conjecture in the two-dimensional case. In [Le], Ledrappier proved the conjecture for Bowen–Ruelle–Sinai measures, and in [PY], Pesin and Yue extended this result to some measures including Gibbs measures for Axiom A diffeomorphisms. A complete proof has recently been announced in [BPS].

In [PW1], we *verify* the Eckmann–Ruelle Conjecture for Gibbs measures for Hölder continuous conformal repellers and conformal Axiom A$^\#$ [topologically hyperbolic (see [AJ])] homeomorphisms. We also construct a Hölder continuous Axiom A$^\#$ homeomorphism of positive topological entropy for which the unique measure of maximal entropy is ergodic and has different upper and lower pointwise dimensions almost everywhere. This example shows that the nonconformal Hölder continuous version of the Eckmann–Ruelle Conjecture is false and thus the smoothness requirement in the conjecture is crucial.

In [C], Cutler constructed an example of a continuous map of [0, 1] that preserves an ergodic measure λ such that $d_\lambda(x)$ exists almost everywhere but is essentially nonconstant. Her example has a zero Lyapunov exponent. In [PW1] the authors present a more refined version of her construction and show that such a map can be arranged to be Hölder continuous and topologically hyperbolic. Again we see the smoothness requirement in the conjecture is crucial.

If a map is smooth and ergodic with respect to a measure λ, then the upper and lower pointwise dimensions of λ are invariant measurable functions, and by the Birkhoff ergodic theorem are constant almost everywhere. Denote these values by $\underline{d}_\lambda$ and $\overline{d}_\lambda$. Ledrappier and Misiurewicz [LM] constructed a one-dimensional smooth (C^r for any $r<\infty$) map preserving an ergodic measure such that $\underline{d}_\lambda<\overline{d}_\lambda$. Obviously the measure λ is not exact dimensional. Their map has a zero Lyapunov exponent. Thus the hyperbolicity hypothesis in the conjecture is crucial.

VI. THE LYAPUNOV SPECTRUM

We first consider the case of conformal expanding maps. Let $g:J\to J$ be a smooth conformal expanding map. We define the *Lyapunov exponent* of g at x by

$$\chi(x)=\lim_{n\to\infty}\frac{1}{n}\log\|dg_x^n\|=\lim_{n\to\infty}\frac{1}{n}\log\prod_{k=0}^{n-1}|a(g^k(x))|,$$

if the limit exists.

Since the map g is expanding, if the above limit exists, then it must be strictly positive. Let ν be an invariant Borel probability measure for g that is supported on J. It follows from the Subadditive Ergodic Theorem that $\chi(x)$ exists for ν-almost every x and defines a ν-measurable function. This function is typically no more regular than just measurable, hence any smoothness result related to $\chi(x)$ is surprising. The function $\chi(x)$ is clearly g-invariant.

Given any invariant measure ν, it follows from the Birkhoff Ergodic Theorem that $\chi(x)$ exists almost everywhere with respect to ν. Thus we obtain a decomposition of the set J by

$$J=\{x\in J|\chi(x)\ \text{does not exist}\}\cup\bigcup_{\beta\in\mathbb{R}^+}\{x\in J|\chi(x)=\beta\}.$$

If the measure ν is ergodic, then $\chi(x)=\chi_\nu=\int_J\log|a(x)|d\nu(x)$ for ν-almost every $x\in J$ and we call χ_ν the *Lyapunov exponent for* ν. We obtain the decomposition of the set J by

$$J=\{x\in J|\chi(x)=\chi_\nu\}\cup\{x\in J|\chi(x)\ \text{does not exist}\}\cup\bigcup_{\substack{\beta\in\mathbb{R}^+\\ \beta\neq\chi_\nu}}\{x\in J|\chi(x)=\beta\}.$$

There are several fundamental questions related to this decomposition. Do there exist points x such that $\chi(x)$ exists but does not equal χ_ν? Since the ν measure of this set is zero, what is the Hausdorff dimension of this set? How large is the set of values attained by $\chi(x)$, for example, does it contain an interval? Do there exist points x such that $\chi(x)$ does not exist, and if so, does the set have a positive Hausdorff dimension? Since Lyapunov exponents are fundamental invariants of a smooth dynamical system, it seems important to have a good understanding of this decomposition.

The following definition for the Lyapunov spectrum was inspired by a paper of Eckmann and Procaccia [EP]. We define the *Lyapunov (exponent) spectrum for the map* g by

$$\ell(\beta)=\dim_H L_\beta, \quad \text{where} \quad L_\beta=\{x\in J \mid \chi(x)=\beta\}.$$

Our strategy consists of first establishing a link between the Lyapunov spectrum and the dimension spectrum and then using results from Secs. II and III about the dimension spectrum to obtain analogous results for the Lyapunov spectrum.

We now describe another characterization of the Lyapunov exponent for conformal expanding maps that allows us to apply some results in [PW1] and relate the Lyapunov exponent at a point to the pointwise dimension at that point. Choose a Markov partition for the map g. As before, consider the basic sets

$$R_{i_1\ldots i_n}=R_{i_1}\cap g^{-1}R_{i_2}\cap\cdots\cap g^{-n+1}R_{i_n},$$

where g^{-i} denotes a branch of the inverse of g^i. By the Markov property, every basic set $R_{i_1\ldots i_n} = R_{i_1}\cap h(R_{i_n})$ for some branch h of g^{-n+1}. Let $R_{i_1\ldots i_n}(x)$ denote a basic set at level n that contains the point x.

An easy argument using the Jacobian estimate shows that the Lyapunov exponent at a point x is the exponential decay rate of the diameter of the basic set that contains x, i.e., the Lyapunov exponent of g at x satisfies

$$\chi(x)=\lim_{n\to\infty}\frac{1}{n}\log\|dg_x^n\|$$

$$=-\lim_{n\to\infty}\frac{1}{n}\log\operatorname{diam}(R_{i_1\ldots i_n}(x)).$$

The following theorem [W1] establishes a relation between the pointwise dimension at a point and the Lyapunov exponent at the point.

Theorem VI.1: Let $g\colon J\to J$ be a smooth conformal expanding map and let ν_ξ be the Gibbs measure corresponding to the Hölder continuous potential ξ. If the Birkhoff average $\lim_{n\to\infty}1/n\sum_{i=0}^{n-1}\xi(g^k(x))\equiv\bar{\xi}(x)$, then

$$d_{\nu_\xi}(x)=\frac{P(\xi)-\bar{\xi}(x)}{\chi(x)}=\frac{h_{\nu_\xi}(g)+\int\xi\, d\nu_\xi-\bar{\xi}(x)}{\chi(x)}, \tag{11}$$

provided that $\bar{\xi}(x)$ and $\chi(x)$ exist, where $P(\xi)$ denotes the thermodynamic pressure of the function ξ (see Appendix A) and $h_{\nu_\xi}(g)$ denotes the measure theoretic entropy of the map g with respect to the measure ν_ξ.

The theorem is straightforward to establish for the symbolic pointwise dimension. One then applies Theorem III.2. ■

Assume the Lyapunov exponent $\chi(x)$ exists at a point x. For an arbitrary Gibbs measure, the numerator in (11) may or may not be defined for this value of x. However, for the measure of maximal entropy, the numerator *always* exists and equals the topological entropy.

We present a few applications of this theorem. The first application exploits this last observation and establishes the link between the Lyapunov spectrum and the dimension spectrum for the measure of maximal entropy.

Let $\nu_{\max}$ denote the *measure of maximal entropy* for g. The measure $\nu_{\max}$ is the Gibbs measure corresponding to a constant potential.

Proposition VI.1: Let $g\colon J\to J$ be a smooth conformal expanding map. Then

(1) If $\nu_{\max}\neq m$ then the Lyapunov spectrum $\ell(\beta) = f_{\nu_{\max}}(h_{\text{TOP}}(g)/\beta)$ is a real analytic strictly convex function on an open interval I containing the point $\beta=d$.
(2) If $\nu_{\max}=m$ then the Lyapunov spectrum $\ell(\beta) = f_{\nu_{\max}}(h_{\text{TOP}}(g)/\beta)$ is a delta function, i.e.,

$$\ell(\beta)=\begin{cases} d, & \text{for } \beta=h_{\text{TOP}}(g)/d,\\ 0, & \text{for } \beta\neq h_{\text{TOP}}(g)/d,\end{cases}$$

where $d=\dim_H J$ and $h_{\text{TOP}}(g)$ is the topological entropy of g.

This immediately implies the following proposition.

Proposition VI.2: Let $g\colon J\to J$ be a smooth conformal expanding map for which $\nu_{\max}\neq m$. Then the range of $\chi(x)$ contains an open interval, and hence the Lyapunov exponent $\chi(x)$ attains innumerable distinct values.

We obtain the following rigidity result as a simple corollary of Proposition VI.2.

Proposition VI.3: Let $g\colon J\to J$ be a smooth conformal expanding map. If the Lyapunov exponent $\chi(x)$ attains only countably many values, then $m=\nu_{\max}$.

Combining this proposition with a theorem of Zdunik [Z], we obtain the following rigidity theorem for rational maps.

Theorem VI.2: If the Lyapunov exponent of a rational map having a hyperbolic Julia set attains only countable values, then the map must be of the form $z\to z^{\pm n}$.

We now consider the Lyapunov spectrum for Axiom A surface diffeomorphisms. Let Λ be a basic set for an Axiom A surface diffeomorphism $f\colon M\to M$. For each $x\in\Lambda$ we have the functions $a^s(x)=-\|df|E^s(x)\|$ and $a^u(x)=\|df|E^u(x)\|$ (see Sec. III). Define the *positive* and *negative Lyapunov exponents* $\chi^+(x)$ and $\chi^-(x)$ by

$$\chi^+(x)=\lim_{n\to\infty}\frac{\log\|df|E^u(x)\|}{n}=\lim_{n\to\infty}\frac{\log\prod_{k=0}^{n-1}a^u(f^k(x))}{n} \tag{12}$$

and

$$\chi^-(x)=\lim_{n\to\infty}\frac{\log\|df|E^s(x)\|}{n}=\lim_{n\to\infty}\frac{\log\prod_{k=0}^{n-1}a^s(f^k(x))}{n},$$

if the limits exist. Since $df|E^u(x)$ is expanding and $df|E^s(x)$ is contracting, if the limits exist they must be nonzero. If ν is an invariant Borel probability measure, it follows from the subadditive ergodic theorem that $\chi^+(x)$ and $\chi^-(x)$ exist for ν almost every x and define f-invariant measurable functions.

Let $L_\beta^+=\{x\in\Lambda|\chi^+(x)=\beta\}$. Consider the following decomposition of the set Λ associated with (positive) values of the Lyapunov exponent $\chi^+(x)$ at points $x\in\Lambda$

$$\Lambda=\{x\in\Lambda|\chi^+(x)\text{does not exist}\}\cup\bigcup_{\beta\in\mathbb{R}^+}L_\beta^+.$$

If ν is an ergodic measure for f, there is a positive constant χ_ν^+ such that $\chi^+(x)=\chi_\nu^+=\int_\Lambda\log a^u(x)d\nu(x)$ for ν-almost every $x\in\Lambda$ and we call χ_ν^+ *the positive Lyapunov exponent for* ν. If ν is the Gibbs measure corresponding to a Hölder continuous function, this set is dense everywhere.

As in the case of conformal repellers, there are several fundamental questions related to the above decomposition. Are there points x for which the limit in (12) exists but does not equal χ_ν^+ for any measure ν? How large is the range of values of $\chi^+(x)$? Are there points x for which the limit in (12) does not exist?

We introduce the *(positive) Lyapunov dimension spectrum* of f by

$$\ell^+(\beta)=\dim_H L_\beta^+.$$

Since the measure of maximal entropy is the unique Gibbs measure corresponding to the function $\varphi=0$ it follows that, $\psi=\text{constant}=\exp(-h_{\text{TOP}}(f))$, where $h_{\text{TOP}}(f)$ is the topological entropy of f on Λ. By slightly modifying the proof of Theorem VI.1 one can show that for *every* $x\in L_\beta^+$

$$d_{\nu_{\max}^u}(x)=\frac{h_{\text{TOP}}(f)}{\beta}.$$

where $d_{\nu_{\max}^u}(x)$ denotes the pointwise dimension of the conditional measure induced by $\nu_{\max}$ on the local unstable manifold passing through x. Let us notice that near a point $x\in\Lambda$, the measure $\nu_{\max}$ is equivalent to the direct product measure $\nu_{\max_x}^s\times\nu_{\max_x}^u$, where $\nu_{\max_x}^s$ denotes the conditional measure induced by $\nu_{\max}$ on the local stable manifold passing through x.

By combining this result with Theorem III.1 we obtain the following result.

Theorem VI.3:

(1) If $\nu_{\max}^u|R(x)$ is not equivalent to the measure $m_x^u|R(x)$ for any $x\in\Lambda$, then the Lyapunov spectrum

$$\ell^+(\beta)=f_{\nu_{\max}^u}\left(\frac{h_{\text{TOP}}(f)}{\beta}\right)$$

is a real analytic strictly convex function on an open interval containing the point $\beta=h_{\text{TOP}}(f)/\dim_H\Lambda$.

(2) If $\nu_{\max}|R(x)$ is equivalent to $m_x^u|R(x)$ for any $x\in\Lambda$, then the Lyapunov spectrum is a delta function, i.e.,

$$\ell^+(\beta)=\begin{cases}\dim_H\Lambda, & \text{for }\beta=h_{\text{TOP}}(f)/\dim_H\Lambda,\\ 0, & \text{for }\beta\neq h_{\text{TOP}}(f)/\dim_H\Lambda.\end{cases}$$

As immediate consequences of this result we obtain that if the measure $\nu_{\max}|R(x)$ is not equivalent to the measure $m_x^u|R(x)$ for any $x\in\Lambda$ then the range of the function $\chi^+(x)$ contains an open interval, and hence, the Lyapunov exponent attains uncountably many distinct values. Hence if the Lyapunov exponent $\chi^+(x)$ attains only countably many values, then $\nu_{\max}|R(x)$ is equivalent to $m_x^u|R(x)$ for any $x\in\Lambda$.

Similar statements hold true for the *negative Lyapunov dimension spectrum* of f corresponding to negative values of the Lyapunov exponent $\chi^-(x)$ at points $x\in\Lambda$.

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APPENDIX A: HAUSDORFF AND BOX DIMENSIONS

Two well-known, dimension-like characteristics of a set $Z\subset\mathbb{R}^n$ are the box dimension and the Hausdorff dimension.

Let $N_\delta(Z)$ denote the minimum number of sets of diameter precisely δ needed to cover the set Z. We define the *lower and upper box dimensions* of Z by

$$\underline{\dim}_B Z=\liminf_{\delta\to 0}\frac{\log N_\delta(Z)}{\log\left(\frac{1}{\delta}\right)},$$

and

$$\overline{\dim}_B Z=\limsup_{\delta\to 0}\frac{\log N_\delta(Z)}{\log\left(\frac{1}{\delta}\right)}.$$

If the numbers $\underline{\dim}_B Z$ and $\overline{\dim}_B Z$ coincide, we denote the common value by $\dim_B Z$ and call it the *box dimension* of the set Z.

It is not hard to show that the box dimension of a set coincides with the box dimension of the closure of the set, and hence the box dimension of a countable dense set contained in $\mathbb{R}^n$ is n.

A *finer* notion of dimension is the Hausdorff dimension. For a fixed $\delta>0$, one considers covers of the set Z by sets of diameter $\leqslant\delta$. For any $s>0$ we define the *s-dimensional Hausdorff measure of* Z by

$$m_H(s,Z)=\lim_{\delta\to 0}\inf_{\{U_i\}}\left\{\sum_i \operatorname{diam}(U_i)^s\,\middle|\,\{U_i\}\text{ is a }\delta\text{-cover of }Z\right\},$$

where $\{U_i\}$ is a countable δ-cover of Z, i.e., a countable cover of Z by sets each having diameter less than or equal to δ. There exists a unique critical value of s at which $m_H(s,Z)$ jumps from ∞ to 0. This critical value is called the Hausdorff dimension of Z and is written $\dim_H Z$.

It is easy to see that $\dim_H Z\leqslant\underline{\dim}_B Z\leqslant\overline{\dim}_B Z$. We believe that for a *typical* set Z the inequalities are strict, i.e., we have that $\dim_H Z<\underline{\dim}_B Z<\overline{\dim}_B Z$. In [PoW], the authors exhibit affine Smale Horseshoes $F\subset\mathbb{R}^3$ such that $\dim_H F<\dim_B F$.

The Hausdorff dimension is determined on the subset of Z having the largest Hausdorff dimension. More precisely, the Hausdorff dimension of a disjoint union of sets is equal to the supremum of the Hausdorff dimensions of the individual sets.

APPENDIX B: THERMODYNAMIC FORMALISM

This appendix contains some essential definitions and facts from symbolic dynamics and thermodynamic formalism. For details consult [B, Ru1]. Let X denote a compact metric space and let $C(X)$ denote the space of real valued continuous functions on X.

(1) Let g: $X\to X$ be a continuous map. We define the *pressure* $P:C(X)\to\mathbb{R}$ defined by

$$P_X(\varphi)=\sup_{\mu\in\mathfrak{M}(X)}\left(h_\mu(g)+\int_X \varphi d\mu\right),$$

where $\mathfrak{M}(X)$ denotes the set of shift-invariant probability measures on X, and $h_\mu(f)$ denotes the measure theoretic entropy of the map g with respect to the measure μ. A Borel probability measure $\mu=\mu_\varphi$ on X is called an *equilibrium measure* for the potential $\varphi\in C(X)$ if

$$P_X(\varphi)=h_\mu(g)+\int_X \varphi d\mu.$$

For any continuous function, an equilibrium measure exists, but may not be unique. It is unique if the function is Hölder continuous. For hyperbolic maps, e.g., conformal repellers or Axiom A diffeomorphisms, equilibrium measures are often called Gibbs measures since the pullback of an equilibrium measure under a coding map (defined using a Markov partition) is Gibbs.

(2) The pressure function $P:C^\alpha(\Sigma_A^+,\mathbb{R})\to\mathbb{R}$ is real analytic. We remark that this result may not be true if Σ_A^+ is replaced by an arbitrary symbolic system.

(3) Let $\varphi\in C^\alpha(\Sigma_A^+,\mathbb{R})$. The map $\mathbb{R}\to\mathbb{R}$ defined by $t\to P(t\varphi)$ is convex. It is strictly convex unless φ is cohomologous to a constant, i.e., there exists $C>0$ and $g\in C^\alpha(\Sigma_A^+,\mathbb{R})$ such that $\varphi(x)=g(\sigma x)-g(x)+C$.

(4) Let $\varphi\in C(\Sigma_A^+)$. A Borel probability measure $\mu=\mu_\varphi$ on Σ_A^+ is called a *Gibbs measure* for the potential φ if there exist constants $D_1,D_2>0$ such that

$$D_1\leqslant\frac{\mu\{y|y_i=x_i,i=0,\ldots,n-1\}}{\exp(-nP(\varphi)+\Sigma_{k=0}^{n-1}\varphi(\sigma^k x))}\leqslant D_2$$

for all $x=(x_1x_2\ldots)\in\Sigma_A^+$ and $n\geqslant 0$. For subshifts of finite type, Gibbs measures exist for any Hölder continuous potential φ, are unique, and coincide with the Gibbs measure for φ.

(5) Given two continuous functions h_1 and h_2 on Σ_A^+ we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}P(h_1+\varepsilon h_2)=\int_J h_2 d\mu_{h_1},$$

where μ_{h_1} denotes the Gibbs measure for the potential h_1.

For $f,g\in C^\alpha(\Sigma_A^+,\mathbb{R})$, the function $t\mapsto P(f+tg)$ is convex. It is strictly convex if and only if g is not cohomologous to a constant.

A. Facts about the Legendre transform

Let f be a C^2 strictly convex map on an interval I, hence, $f''(x)>0$ for all $x\in I$. The *Legendre transform* of f is the function g of a new variable p defined by

$$g(p)=\max_{x\in I}(px-f(x)).$$

It is easy to show that g is strictly convex and that the Legendre transform is involutive. One can also show that strictly convex functions f and g form a Legendre transform pair if and only if $g(\alpha)=f(q)+q\alpha$, where $\alpha(q)=-f'(q)$ and $q=g'(\alpha)$. See [Ro] for more details.

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