OPEN PROBLEMS IN THE THEORY OF NON-UNIFORM HYPERBOLICITY

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Abstract. This is a survey-type article whose goal is to review some recent developments in studying the genericity problem for non-uniformly hyperbolic dynamical systems with discrete time on compact smooth manifolds. We discuss both cases of systems which are conservative (preserve the Riemannian volume) and dissipative (possess hyperbolic attractors). We also consider the problem of coexistence of hyperbolic and regular behaviour.

1. Introduction

How prevalent is deterministic chaos? It has been understood since the 1960s that a deterministic dynamical system can exhibit apparently stochastic behaviour. This is due to the fact that instability along typical trajectories of the system, which drives orbits apart, can coexist with compactness of the phase space, which forces them back together; the consequent unending dispersal and return of nearby trajectories is one of the hallmarks of chaos.

Of course, not every dynamical system exhibits such instability; there are many systems whose behaviour is quite regular and not at all chaotic. Thus it is natural to ask which sort of behaviour prevails: is regularity the rule, and chaos the exception? Or is it the other way around? Perhaps there are different contexts in which either sort of behaviour is "typical". Many of the open problems regarding chaotic systems at the present time are related to this question.

In order to meaningfully address this issue, a number of things need to be made precise. What exactly do we mean by "chaos", and what sort of "instability" do we consider? What do we mean by a "typical" dynamical system, and what does it mean for one sort of behaviour to be the "rule", and the other the "exception"?

The first of these questions has multiple potential answers. The common thread in all of them is that when f is a diffeomorphism of a smooth Riemannian manifold M, instability of trajectories $f^n(x)$ manifests itself as a splitting of the tangent spaces $T_{f^n(x)}M$ into invariant subspaces along which f has prescribed expansion and contraction rates. This splitting allows us to

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model the behaviour of nearby trajectories of f on the behaviour of trajectories in the neighbourhood of a hyperbolic fixed point. There are a number of possible ways in which this splitting can occur, leading to various classes of dynamical systems; these can be described by appropriate *hyperbolicity conditions* (see the next section).

One problem of great importance in the modern theory of dynamical systems is to determine whether hyperbolic dynamical systems are generic in some sense. The goal of this paper is to review some recent developments and to discuss some remaining open problems related to genericity of hyperbolicity and hence to prevalence of chaotic behavior.

We restrict ourselves to the case of dynamical systems with discrete time, where the problem can be viewed in terms of a special "derivative" cocycle over a diffeomorphism. The genericity problem can be extended to include all linear cocycles over a given diffeomorphism, a situation that is better understood and for which many interesting results have been obtained (see for example, [BP05]); however, these advances do not cover the case of derivative cocycles.

2. Hyperbolicity conditions

For a complete exposition of all the topics in this section, see [BP07]. The most restrictive form of hyperbolicity (which is also the most well understood) is the case of *uniform complete hyperbolicity*, when every tangent space admits a splitting

$$T_x M = E^s(x) \oplus E^u(x) \tag{2.1}$$

such that f is uniformly contracting on the *stable subspace* $E^s(x)$ and uniformly expanding on the *unstable subspace* $E^u(x)$, and the splitting is f-invariant. Formally, a compact f-invariant set Λ is *hyperbolic* for f if

1. the tangent space has a splitting (2.1) at every $x \in \Lambda$ such that $Df(E^s(x)) = E^s(f(x))$ and $Df(E^u(x)) = E^u(f(x))$;
2. there exist constants $C > 0$ and $\lambda \in (0,1)$ such that for all $x \in \Lambda$ and $n \geq 1$, we have

$$\begin{aligned} \|Df^n v\| &\leq C\lambda^n \|v\| \text{ for all } v \in E^s(x), \\ \|Df^{-n} v\| &\leq C\lambda^n \|v\| \text{ for all } v \in E^u(x). \end{aligned} \tag{2.2}$$

If the entire manifold M is a hyperbolic set, we say that f is *Anosov*; if the non-wandering set $\Omega(f) \subset M$ is a hyperbolic set in which periodic orbits are dense, we say that f is *Axiom A*.

Anosov and Axiom A systems have very strong chaotic properties, and a great deal is known about their behaviour. In particular, the stable and unstable subspaces form two continuous (in fact, Hölder continuous) subbundles of the tangent bundle each of which is integrable to a continuous foliation of M with smooth leaves. Thus we obtain two transversal foliations of M, into *stable* manifolds $W^s(x)$ and *unstable* manifolds $W^u(x)$. By the classical Hadamard–Perron theorem, the global leaves of these foliations can be characterized as follows

$$W^s(x) = \{y \in M \mid d(f^n(y), f^n(x)) \to 0,\ n \to \infty\},$$
$$W^u(x) = \{y \in M \mid d(f^n(y), f^n(x)) \to 0,\ n \to -\infty\}$$

(in fact, the convergence is exponential). Let $B(x, r)$ be the ball centred at x of radius r: for $x \in M$ and a sufficiently small number $r > 0$, the connected component of the intersection $W^s(x) \cap B(x, r)$ (respectively, $W^u(x) \cap B(x, r)$) is a *local stable leaf* (respectively, a *local unstable leaf*) of the foliation through x, denoted $V^s(x)$ (respectively, $V^u(x)$).

The stable and unstable foliations satisfy the crucial *absolute continuity property*: given a set of positive volume, its intersection with almost every leaf of either foliation has positive leaf-volume. Indeed, given a point $x \in M$, consider the partition ξ^s (respectively, ξ^u) of the ball $B(x, r)$ of small radius r around x by local stable leaves $V^s(y)$ (respectively, local unstable leaves $V^u(y)$), $y \in B(x, r)$; then for almost every y, the conditional measure generated by volume on the leaf $V^s(y)$ (respectively, $V^u(y)$)) is equivalent to the leaf-volume (i.e., the Riemannian volume on the leaf generated by the Riemannian metric) with bounded and strictly positive density.

The condition of uniform complete hyperbolicity is too strong to account for the diverse variety of chaotic behaviour which is observed in various systems. Thus, weaker versions of hyperbolicity are needed.

A more general class of systems are those displaying *non-uniform complete hyperbolicity*, in which the constants C and λ in (2.2) are no longer assumed to be independent of the point x; we allow them to vary from one point to another, but control how they change along an orbit of f. Formally, an f-invariant set Y (which need not be compact) is *non-uniformly completely hyperbolic* if the tangent space at every $x \in Y$ has an invariant splitting (2.1) that depends measurably on x, and that satisfies the following in place of (2.2):

(1) There exist positive Borel functions $C, K, \varepsilon, \lambda \colon Y \to (0, \infty)$ such that $\lambda(x)e^{\varepsilon(x)} < 1$ for all $x \in Y$ and such that for every $n \geq 1$, we have

$$\begin{aligned} \|Df^n v\| &\leq C(x)\lambda(x)^n e^{\varepsilon(x)n}\|v\| \text{ for all } v \in E^s(x), \\ \|Df^{-n} v\| &\leq C(x)\lambda(x)^n e^{\varepsilon(x)n}\|v\| \text{ for all } v \in E^u(x). \end{aligned} \tag{2.3}$$

(2) λ and ε are f-invariant, and the functions C and K vary slowly along trajectories:

$$C(f^n(x)) \leq C(x)e^{\varepsilon(x)|n|}, \quad K(f^n(x)) \geq K(x)e^{-\varepsilon(x)|n|} \quad \text{for all } n \in \mathbb{Z}. \tag{2.4}$$

(3) The angle between the subspaces $E^s(x)$ and $E^u(x)$ satisfies

$$\angle(E^s(x), E^u(x)) \geq K(x). \tag{2.5}$$

We stress that the set Y need not be compact. Indeed, if a non-uniformly completely hyperbolic set Y is compact then in fact, Y is uniformly completely hyperbolic (see [HPS07]).

A useful tool in constructing non-uniformly completely hyperbolic sets is the *Lyapunov exponent*

$$\chi^+(x,v) = \overline{\lim_{n\to+\infty}} \frac{1}{n} \log \|Df^n v\|. \tag{2.6}$$

For any small $\varepsilon > 0$ and large positive $n = n(\varepsilon)$, (2.6) implies that

$$\|Df^n v\| \approx e^{(\chi^+(x,v)\pm\varepsilon)n},$$

and so a natural candidate for the stable subspace $E^s(x)$ is

$$\{v \in T_xM \mid \chi^+(x,v) < 0\}. \tag{2.7}$$

Similarly, reversing the time we obtain the backwards Lyapunov exponent

$$\chi^-(x,v) = \overline{\lim_{n\to-\infty}} \frac{1}{n} \log \|Df^n v\|, \tag{2.8}$$

so that for any small $\varepsilon > 0$ and large negative $n = n(\varepsilon)$, we have

$$\|Df^n v\| \approx e^{(\chi^-(x,v)\pm\varepsilon)n}.$$

Therefore a natural candidate for the unstable subspace $E^u(x)$ is

$$\{v \in T_xM \mid \chi^-(x,v) < 0\}. \tag{2.9}$$

If one knows that $\chi^+(x,v) = -\chi^-(x,v)$, then the natural candidates (2.7) and (2.9) span the tangent space T_xM if and only if f has no zero Lyapunov exponents at the point x—that is, if

$$\chi^+(x,v) \neq 0 \text{ for all } v \in T_xM. \tag{2.10}$$

To guarantee that these natural candidates actually satisfy the definition of non-uniform complete hyperbolicity once (2.10) holds, one needs a further condition, called *Lyapunov–Perron regularity*, that Df must satisfy along the orbit of x. The Multiplicative Ergodic Theorem of Oseledets guarantees that this condition holds μ-a.e., where μ is any invariant Borel measure. Thus non-uniform complete hyperbolicity may be restated in terms of Lyapunov exponents as a property of a system *and an invariant measure*.

An invariant Borel measure μ is said to be *hyperbolic* if (2.10) holds for μ-a.e. $x \in M$. One can show that if μ is a hyperbolic probability measure for f, then there exists a non-uniformly completely hyperbolic set Y such that $\mu(Y) = 1$. For every $x \in Y$, one can construct the global stable, $W^s(x)$ (and respectively, the global unstable, $W^u(x)$) manifold through x. It consists of points whose trajectories converge with an exponential rate to the trajectory of x when time $n \to +\infty$ (respectively, $n \to -\infty$).

The global stable (respectively, unstable) manifolds form what one can regard as a "measurable" foliation with smooth leaves. A better description of this foliation can be obtained as follows. First, there exist compact (but non-invariant) sets Y_ℓ (called *Pesin sets*) for $\ell \in \mathbb{N}$, which have the following properties:

1. the sets Y_ℓ are nested ($Y_\ell \subset Y_{\ell+1}$);
2. they exhaust a full measure set ($Y = \bigcup_\ell Y_\ell \pmod 0$);

(3) each set Y_ℓ is uniformly hyperbolic: the estimates (2.3) and (2.5) are uniform on each Y_ℓ. More precisely, there exist constants $C = C(\ell)$ and $K = K(\ell)$ such that $C(x) \le C$ and $K(x) \ge K$ for all $x \in Y_\ell$; these constants may deteriorate with ℓ (i.e., $C = C(\ell)$ may be unbounded and $K = K(\ell)$ may approach zero as $\ell \to \infty$).

Now for every ℓ, there exists $r = r(\ell) > 0$ such that at each point $x \in Y_\ell$, one has a *local* stable manifold $V^s(x)$ with the following properties:

(1) $V^s(x)$ is the connected component of the intersection $W^s(x) \cap U$, where $U \subset M$ is a small neighbourhood of x;
(2) Using local coordinates near x that come from the decomposition $T_xM = E^s(x) \oplus E^u(x)$, there is a ball $B^s(x,r) \subset E^s(x)$ and a function $\psi\colon B^s(x,r) \to E^u(x)$ whose graph is $V^s(x)$.

Similar properties hold for the local unstable manifolds $V^u(x)$. The quantity $r(\ell)$ may be thought of as the "size" of the local stable and unstable manifolds on the Pesin set Y_ℓ.

On each Y_ℓ, the family of local stable leaves satisfies the absolute continuity property. More precisely, consider the partition ξ^u of the set

$$Q^s(x) = \bigcup_{y \in Y_\ell \cap B(x,r)} V^u(y) \tag{2.11}$$

by local unstable leaves $V^u(y)$; then the conditional measure generated by volume on almost every leaf $V^u(y)$ is equivalent to the leaf-volume with bounded and strictly positive density. Similarly, the family of local stable leaves satisfies the absolute continuity property.

The condition of complete hyperbolicity can also be generalized by replacing (2.1) with a splitting of the form

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x), \tag{2.12}$$

where $E^s(x)$ and $E^u(x)$ are as before, and f may be either expanding or contracting (or both) on the *central subspace* $E^c(x)$; the only requirement is that the rate of expansion or contraction on $E^c(x)$ be less than the corresponding rates on $E^s(x)$ and $E^u(x)$. Such systems are called *partially hyperbolic*. As with complete hyperbolicity, partial hyperbolicity can be either uniform or non-uniform.

For a uniformly partially hyperbolic system, the stable and unstable subbundles are continuous and are integrable to continuous foliations W^s and W^u with smooth leaves. These foliations satisfy the absolute continuity property. The central subbundle may or may not be integrable and often fails to satisfy the absolute continuity property.

A weaker version of uniform partial hyperbolicity is the existence of a *dominated splitting*: that is, an invariant splitting $T_xM = E_1(x) \oplus E_2(x)$ for which there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|_{E_1(x)}\| \cdot \|Df^{-n}|_{E_2(f^n(x))}\| \le C\lambda^n \tag{2.13}$$

for all $x \in M$ and $n \in \mathbb{N}$. Observe that neither of the distributions E_1, E_2 is required to be uniformly stable or unstable; it is entirely possible that E_1 is contracting at some points and in some directions, and expanding elsewhere, and similarly for E_2. All that is required is that along any given orbit, every vector in E_2 is asymptotically expanded more than every vector in E_1.

For a uniformly partially hyperbolic system, the splitting (2.12) is dominated, but by the remarks in the previous paragraph, the existence of a dominated splitting does not imply uniform partial hyperbolicity. It *does* imply non-uniform partial hyperbolicity, which amounts to requiring that *some* (not necessarily all) Lyapunov exponents be non-zero, with some being positive and some negative.

3. Conservative and dissipative systems

Of course, any given dynamical system usually has many invariant measures, and may well have many hyperbolic measures. A question of particular interest is whether or not f has an invariant measure which is absolutely continuous with respect to volume on M—such systems are called *conservative*. In this case the measure of the most interest is the absolutely continuous invariant measure.

There are also many systems of interest that do not preserve such a measure—such systems are called *dissipative*. The starting point for analysis of a dissipative system is a *trapping region*—an open set U such that $\overline{f(U)} \subset U$. The set of points that have an infinite string of pre-images in U form an *attractor* $\Lambda = \bigcap_{n\geq 0} f^n(U)$: this set typically has zero volume and a fractal structure.

The set Λ is compact, and if $f|\Lambda$ is uniformly completely (respectively, partially) hyperbolic, then Λ is called a *hyperbolic attractor* (respectively, a *partially hyperbolic attractor*). It is easy to show that $W^u(x) \subset \Lambda$ for every $x \in \Lambda$; thus the fractal structure of Λ appears in the directions transversal to the unstable direction. More precisely, in a small neighborhood $B(x, r)$ of a point $x \in \Lambda$, the intersection $B(x, r) \cap \Lambda$ is homeomorphic to $C \times V^u(x)$, where C is a Cantor set and $V^u(x)$ a local unstable manifold at x.

Every invariant measure in U is supported on Λ, and hence is singular with respect to volume; thus a new technique for selecting a measure of interest is required. For a completely hyperbolic attractor, one can construct a special hyperbolic measure μ that is absolutely continuous along unstable manifolds—that is, the conditional measures generated by μ on local unstable leaves are equivalent to the leaf-volume. Such a measure μ is called an *SRB measure* (after Sinai, Ruelle, and Bowen), and it has many good ergodic properties. If $f|\Lambda$ is topologically transitive there is only one SRB measure for f on Λ. SRB measures are *physical* in the following sense: writing δ_x for the point measure sitting at x, the *basin of attraction* of a measure μ is the set

$$B(\mu) = \left\{ x \in M \,\middle|\, \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} = \mu \right\}, \tag{3.1}$$

where convergence is in the weak* topology. The basin of attraction of an SRB measure has full measure in the topological basin of attraction of Λ.

Another characterization of SRB measures that reveals their physical meaning is as follows. Starting from a measure m that is supported on a neighborhood U of Λ and that is equivalent to the Riemannian volume, consider its evolution under the dynamics, i.e., the sequence of measures

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k m. \tag{3.2}$$

The weak* limit of this sequence of measures is the unique SRB measure on Λ.

For partially hyperbolic attractors, one can construct an analog of SRB measures—the so-called u-measures, which are characterised by the fact that their conditional measures on local unstable leaves are equivalent to the leaf-volume—and one can obtain a good deal of information on their ergodic properties [PS82]. In contrast, the concept of non-uniformly hyperbolic attractors is not well understood, and only a handful of examples of non-uniformly hyperbolic attractors are known. We will give a more detailed discussion of partially and non-uniformly hyperbolic attractors in Section 7.

4. Genericity conjectures I: Mixed hyperbolicity

Uniformly (completely or partially) hyperbolic diffeomorphisms form an open set in the space of diffeomorphisms of class C^r for $r \geq 1$. However, the existence of a uniformly completely hyperbolic diffeomorphism, and even of a uniformly partially hyperbolic diffeomorphism, places strong conditions on the topology of the underlying manifold M: there are many compact Riemannian manifolds which do not admit *any* uniformly completely hyperbolic diffeomorphisms. In particular, it is conjectured that if a manifold carries an Anosov diffeomorphism than its universal cover is a Euclidean space, and it is known that a 3-dimensional sphere does not admit a partially hyperbolic diffeomorphism (see [BBI04]). See also [Mar67] for a somewhat related result on the time-one maps of Anosov flows on 3-dimensional manifolds.

Consequently, if we hope to find some sort of chaotic behaviour in "typical" systems, we must rely on the non-uniformly hyperbolic systems introduced in the previous section; we may take heart from the fact that there are no topological obstructions to existence of a non-uniformly completely hyperbolic diffeomorphism.

Theorem 4.1 (see [DP02])**.** *Any compact smooth Riemannian manifold M of dimension ≥ 2 admits a C^∞ volume preserving diffeomorphism f, which has nonzero Lyapunov exponents almost everywhere and is Bernoulli.*

A similar result for flows (in dimension ≥ 3) was proved in [HPT04].

Given a compact smooth Riemannian manifold M, let $\mathrm{Diff}^{1+\alpha}(M)$ denote the space of all $C^{1+\alpha}$ diffeomorphisms on M. We will first consider conservative systems, and so we let μ denote a measure equivalent to volume and write $\mathrm{Diff}^{1+\alpha}(M,\mu)$ for the space of all $C^{1+\alpha}$ diffeomorphisms which preserve μ.

Let $\mathcal{H} \subset \mathrm{Diff}^{1+\alpha}(M,\mu)$ be the set of all $C^{1+\alpha}$ diffeomorphisms f that preserve μ and have non-zero Lyapunov exponents on a set of positive volume. Such systems exhibit non-uniform complete hyperbolicity on a non-negligible part of phase space.

The following conjecture is stated in [Pes07, BP05, BP07], and relates to questions that were first asked in [Pes77].

Conjecture 4.2. *$\mathcal{H}$ is dense in* $\mathrm{Diff}^{1+\alpha}(M,\mu)$.

The regularity assumption in Conjecture 4.2 is crucial; the analogous statement for C^1 diffeomorphisms is false. Indeed, if M is any compact surface, then it was shown by Bochi that there is a residual set $\mathcal{R} \subset \mathrm{Diff}^1(M,\mu)$ such that every $f \in \mathcal{R}$ is either Anosov or has zero Lyapunov exponents almost everywhere. Since Anosov diffeomorphisms are not dense in $\mathrm{Diff}^1(M,\mu)$, the analogue of Conjecture 4.2 fails here.

In higher dimensions, Bochi and Viana [BV05] showed that there is a residual set $\mathcal{R} \subset \mathrm{Diff}^1(M,\mu)$ such that for $f \in \mathcal{R}$ and μ-a.e. $x \in M$, either all Lyapunov exponents vanish at x or there is a dominated splitting along the orbit of x (a *local* dominated splitting). In [AB09], Avila and Bochi proved the even stronger result that for C^1-generic f, one of the following two cases holds: either μ-a.e. point has at least one zero Lyapunov exponent, or there exists a *global* dominated splitting. Since there are topological obstructions to the existence of such splittings on certain manifolds (such as even-dimensional spheres), this shows once again that the analogue of Conjecture 4.2 fails on such manifolds.

In the symplectic case, Bochi and Viana showed that generic C^1 symplectomorphisms are either Anosov or have at least two zero Lyapunov exponents a.e. Avila and Bochi showed that among the Anosov cases, ergodicity is generic, and Avila, Bochi, and Wilkinson [ABW09] showed that ergodicity is C^1-generic among all *partially hyperbolic* symplectomorphisms.

The above results highlight the requirement in Conjecture 4.2 that we work in the $C^{1+\alpha}$ category. (We observe, however, that [AB09] contains several $C^{1+\alpha}$-generic results as well.) As an example of what happens in the higher regularity setting, we consider the following construction, due to Shub and Wilkinson [SW00], which is emblematic of a general situation in which a map with some zero Lyapunov exponents can be perturbed slightly to obtain a map in $\mathcal{H}$.

Example 4.3. Let f be an Anosov diffeomorphism of the 2-dimensional torus $\mathbb{T}^2$, and let F be the direct product map

$$F = (f, \mathrm{Id})\colon \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1. \tag{4.1}$$

Then F is volume preserving and uniformly partially hyperbolic: the central distribution is integrable, with compact leaves of the form $\{x\} \times S^1$ for $x \in \mathbb{T}^2$. F is not ergodic, since every torus $\mathbb{T}^2 \times \{y\}$ is invariant, $y \in S^1$. Furthermore, F has zero Lyapunov exponent in the central direction, and hence $F \notin \mathcal{H}$.

Shub and Wilkinson showed that there exists a volume preserving non-uniformly completely hyperbolic C^∞ diffeomorphism G that is arbitrarily close to F in the C^1 topology. G can be constructed by combining F with a map which is localized in a neighborhood of some point and is a small rotation in the center-unstable direction (see [BP07, §11.2]).

More precisely, fix a point $x \in \mathbb{T}^2$ and consider a disc $D \subset \mathbb{T}^2$ centred at x of a sufficiently small radius $r_0 > 0$. Fix $\varepsilon_0 > 0$, and on the open domain $\Omega = D \times (-\varepsilon_0, \varepsilon_0) \subset \mathbb{T}^3$, consider a coordinate system (x_1, x_2, y), where x_1 and x_2 run over stable and unstable lines of A respectively, $y \in (-\varepsilon_0, \varepsilon_0)$, and $x = (0, 0, 0)$. Choose small positive numbers r_0 and ε and two nonnegative C^∞ bump functions $\rho = \rho(r)$, $0 < r < r_0$ and $\psi = \psi(y)$, $y \in (-\varepsilon_0, \varepsilon_0)$ satisfying

(1) $\rho(r) > 0$ if $0.2r_0 \le r \le 0.8r_0$ and $\rho(r) = 0$ if $0 < r \le 0.1r_0$ or $0.9r_0 \le r \le r_0$;
(2) $\psi(y) = \psi_0$ if $|y| \le 0.8\varepsilon_0$, where $\psi_0 > 0$ is a constant, and $\psi(y) = 0$ if $|y| \ge 0.9\varepsilon_0$.

For a number $\tau > 0$, define the map h_τ on Ω by

$$\begin{aligned} &h_\tau(x_1, x_2, y) \\ &\qquad = (x_1, x_2 \cos(2\pi\tau\alpha) - y \sin(2\pi\tau\alpha), x_2 \sin(2\pi\tau\alpha) + y \cos(2\pi\tau\alpha)) \end{aligned} \tag{4.2}$$

with $\alpha(x_1, x_2, y) = \rho(|x_1|)\psi(\sqrt{x_2^2 + y^2})$, and set $h_\tau = \mathrm{Id}$ on $M \setminus \Omega$. For suitably chosen ρ and ψ, and for every sufficiently small τ, the map h_τ is a C^∞ volume preserving diffeomorphism of M and the perturbation $G_\tau = F \circ h_\tau$ has nonzero Lyapunov exponents on a set of positive volume.

In fact, Shub and Wilkinson showed that if $\tilde{G}$ is *any* volume preserving C^2 diffeomorphism that is sufficiently close to G in the C^1 topology, then $\tilde{G}$ is non-uniformly completely hyperbolic. This provides an open set of non-uniformly completely hyperbolic diffeomorphisms on the 3-torus that are not Anosov.

Their argument was later adapted to the general setting of a (uniformly) partially hyperbolic diffeomorphism with one-dimensional central direction [Dol00, BMVW04]. In this setting, a volume preserving diffeomorphism with zero Lyapunov exponent in the one-dimensional central direction can be perturbed slightly (by a map similar to h_τ in (4.2)) so that it remains partially hyperbolic, and the Lyapunov exponent in the central direction becomes negative (or positive) on a set of positive volume. Furthermore, the set of such perturbations is open, which gives some ground for the following conjecture.

Conjecture 4.4. *If $f \in \mathrm{Diff}^{1+\alpha}(M,\mu)$ has non-zero Lyapunov exponents μ-a.e., then there exists an open neighbourhood $U \subset \mathrm{Diff}^{1+\alpha}(M,\mu)$ of f such that the set $\mathcal{H} \cap U$ is residual in U.*

The result by Shub and Wilkinson is a manifestation of the phenomenon known as *mixed hyperbolicity.* Given a C^2 partially hyperbolic diffeomorphism f preserving a smooth measure μ, we say that f has *negative (positive) central exponents* if there exists an invariant set A of positive μ-measure such that the Lyapunov exponent $\chi^+(x,v)$ is negative (positive) for all $x \in A$ and $v \in E^c(x)$. Thus mixed hyperbolicity amounts to the presence of both partial hyperbolicity and non-uniform hyperbolicity on A, with the added stipulation that the central Lyapunov exponents all take the same sign. (Such maps are said to have a *mostly contracting* or *mostly expanding* central direction.)

In studying the ergodic properties of a system with mixed hyperbolicity, one has the luxury of using methods from both partial and non-uniform hyperbolicity theories. We illustrate this by describing one of the most advanced results in this direction.

We say that two points $x, y \in M$ are *accessible* if they can be connected by a sequence of stable and unstable manifolds—that is, if there exists a collection of points $z_1, \dots, z_n$ such that $x = z_1$, $y = z_n$, and $z_k \in W^s(z_{k-1})$ or $z_k \in W^u(z_{k-1})$ for $k = 2, \dots, n$. Accessibility is an equivalence relation. We say that f has the *accessibility property* if there is only one accessibility class—that is, if any two points are accessible. Furthermore, we say that f has the *essential accessibility property* if any accessibility class has either measure one or zero (with respect to μ).

Theorem 4.5 (see [BDP02])**.** *Assume that f is (essentially) accessible and has negative (positive) central exponents. Then f is ergodic and in fact, is Bernoulli.*

One can strengthen this result by showing that f is *stably ergodic* (indeed, *stably Bernoulli*): any sufficiently small (in the C^1 topology) C^2 perturbation of f, preserving μ, is ergodic (indeed, Bernoulli). Furthermore, any such perturbation has negative (positive) central exponents on the whole phase space.

What about the case where the central direction contains both positive *and* negative Lyapunov exponents? In this direction, a result similar to Theorem 4.5 has been recently obtained in [HHTU07], under the assumption that f is (essentially) accessible, $\dim E^c = 2$, and the Lyapunov exponents in the central subspace are nonzero and of different signs. This motivates the following open problem.

Problem 4.6. *Let f be a C^2 diffeomorphism preserving a smooth measure μ with nonzero central exponents (of which some are positive and some are negative) on a set of positive measure. Assume that f is (essentially) accessible. Is μ ergodic? Is f stably ergodic?*

Let us return now to the general idea of finding non-uniformly hyperbolic diffeomorphisms near a partially hyperbolic C^2 diffeomorphism f with zero central exponents. As mentioned above, this can be done whenever the central direction is one-dimensional: what if $\dim E^c > 1$? In this setting, Baraviera and Bonatti [BB03] showed that f can be perturbed by a map similar to h_τ so that it remains partially hyperbolic, and the *average* Lyapunov exponent in the central direction

$$\chi^c(f) = \int_M \chi^c(f,x)\, d\mathrm{Leb}(x), \text{ where } \chi^c(f,x) = \sum \chi_i^c(x)$$

becomes negative (here $\chi_i^c(x)$ are the distinct values of the Lyapunov exponent for vectors in the center subspace $E^c(x)$ and Leb is volume).

Problem 4.7. *In the above setting, is it possible to perturb f slightly to obtain a volume preserving C^2 diffeomorphism g with negative central exponents?*

Consider again the perturbation G in Example 4.3. The central foliation for G consists of closed one-dimensional smooth curves (which are diffeomorphic to circles); it is continuous (in fact, Hölder continuous) but is *not* absolutely continuous. Moreover, for almost every such curve the conditional measure on it generated by volume is atomic and has exactly one atom (see [RW01]). In other words, there is a set of full measure that intersects almost every leaf of the central foliation in exactly one point: this highly pathological phenomenon, known as "Fubini's nightmare", persists under small perturbations (since any such perturbation has negative central exponents).

This is a reflection of the more general fact that negative Lyapunov exponents cannot coexist with absolute continuity of the central foliation with compact smooth leaves. We present two results supporting this observation.

Consider a C^2 diffeomorphism f of a compact smooth Riemannian manifold M preserving a smooth measure μ and a continuous foliation W of M with smooth compact leaves, which is invariant under f. Let $\chi_W(x)$ denote the sum of the Lyapunov exponents of f at the point x along the subspace $T_xW(x)$. We say that f is *W-dissipative* if there exists an invariant set A of positive μ-measure such that $\chi_W(x) \neq 0$ for μ-almost every $x \in A$.

Theorem 4.8 ([HP07])**.** *If f is W-dissipative almost everywhere then the foliation W is not absolutely continuous.*

Observe that if $\chi^c(f,x) < 0$ for μ-a.e. $x \in M$ then f is W^c-dissipative and the above result applies. The W^c-dissipativity property is typical in the following sense: if $\chi^c(f,x) < -\alpha$ for some $\alpha > 0$ and μ-a.e. $x \in M$, then any diffeomorphism g that is sufficiently close to f in the C^1 topology is W^c-dissipative on a set of positive μ-measure.

If the leaves of the central foliation are not compact, then the following version of the above result holds.

Theorem 4.9 ([SX08]). *Let f be a partially hyperbolic C^2 diffeomorphism preserving a smooth measure μ. Then the central foliation W^c is not absolutely continuous provided $\chi^c(f) > \chi$, where*

$$\chi = \sup_{x\in M} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathrm{Leb}^u(f^n(B^c(x,r)))$$

is the asymptotic growth rate of the leaf-volume Leb^u of the ball $B^c(x,r)$ in W^c centered at x of radius r (and χ does not depend on r).

5. Genericity conjectures II: The Katok map

We will describe another example of a non-uniformly hyperbolic diffeomorphism, this time on the 2-torus, known as the *Katok map*; it was introduced in [Kat79]. This map is a starting point in the construction of non-uniformly hyperbolic diffeomorphisms on arbitrary manifolds (see Theorem 4.1).

Example 5.1. Given $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ with eigenvalues $\lambda^{-1} < 1 < \lambda$, the map $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ is uniformly hyperbolic. Let D_r denote the disc of radius r centred at $(0,0)$, where $r > 0$ is small, and put coordinates (s_1, s_2) on D_r corresponding to the eigendirections of A—that is, $A(s_1,s_2) = (\lambda s_1, \lambda^{-1} s_2)$. Then A is the time-1 map of the flow generated by

$$\dot{s}_1 = s_1 \log\lambda, \qquad \dot{s}_2 = -s_2 \log\lambda. \tag{5.1}$$

The Katok map is obtained from A by slowing down these equations near the origin. This is done by fixing a small value of $r_0 > 0$ and considering a function $\psi\colon [0,1] \to [0,1]$ with the following properties:

(1) ψ is C^∞ except at 0.
(2) $\psi(0) = 0$ and $\psi(u) = 1$ for $u \geq r_0$.
(3) $\psi'(u) > 0$ for all $0 < u < r_0$.
(4) The integral $\int_0^1 \frac{1}{\psi(u)}\,du$ is finite.

Now choosing $0 < r_0 < r_1 < 1$, we let g be the time-1 map of the flow generated by the following slowed-down version of (5.1):

$$\dot{s}_1 = s_1\psi(s_1^2 + s_2^2)\log\lambda, \qquad \dot{s}_2 = -s_2\psi(s_1^2+s_2^2)\log\lambda. \tag{5.2}$$

From the construction of ψ, we have

(1) $g(D_{r_2}) \subset D_{r_1}$ for some $0 < r_2 < r_0 < r_1$;
(2) g is C^∞ in $D_{r_1} \setminus \{0\}$;
(3) g coincides with A in a neighbourhood of the boundary ∂D_{r_1}.

This allows us to define a homeomorphism of the torus by the piecewise formula

$$G(x) = \begin{cases} A(x) & x \in \mathbb{T}^2 \setminus D_{r_1}, \\ g(x) & x \in D_{r_1}. \end{cases} \tag{5.3}$$

In fact, G is a C^∞ diffeomorphism everywhere except for the origin. Although G is no longer uniformly hyperbolic, it can be shown to have non-zero Lyapunov exponents; by smoothing the map out near the origin to remove the point of non-differentiability, one can obtain a genuine non-uniformly hyperbolic system.

To do this, one first observes that G preserves a probability measure ν that is absolutely continuous with respect to Lebesgue measure μ on $\mathbb{T}^2$. Furthermore, via a suitable coordinate change $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$, one can obtain a map $f = \phi \circ G \circ \phi^{-1}\colon \mathbb{T}^2 \to \mathbb{T}^2$ that preserves μ itself, and it turns out that f is C^∞ at the origin as well. In short, one obtains an area-preserving C^∞ diffeomorphism of the 2-torus that has non-zero Lyapunov exponents Lebesgue-a.e.

Furthermore, every point on the image of the stable and unstable eigenlines under the coordinate change ϕ can be shown to have a zero Lyapunov exponent, so f fails to be uniformly hyperbolic.

Problem 5.2. *Prove Conjecture 4.4 for the Katok map.*

It is not difficult to show that any *gentle* perturbation of $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ (that is, a perturbation supported away from the origin) has non-zero Lyapunov exponents; however, gentle perturbations are not generic.

Before moving on, we remark that the requirement in Conjecture 4.4 that f have Hölder continuous derivatives is crucial. Indeed, without the requirement of Hölder continuity, it was shown by Mañé and Bochi [Mañ84, Mañ96, Boc02] that in the two-dimensional case, where M is a surface, a residual set of maps $\mathrm{Diff}^1(M,\mu)$ are either Anosov or have all Lyapunov exponents equal to zero almost everywhere. A similar but weaker result in higher dimensions (involving the notion of *dominated splitting*) was obtained by Bochi and Viana [BV05].

6. Coexistence of chaotic and regular behaviour

We draw special attention to the fact that systems in the class $\mathcal{H}$ are not required to exhibit chaotic behaviour on a set of full measure, but only of *positive* measure. This is because there are many cases in which the phase space of a system can be decomposed into a chaotic part and a regular part, both of positive measure. The regular part, which contains the so-called "elliptic islands", is often persistent under small perturbations of the system, and so cannot be neglected in our search for the behaviour of typical systems.

One case in which regular and chaotic behaviour coexist arises from KAM theory, which guarantees that C^r-small perturbations of integrable Hamiltonian systems have a positive volume Cantor set of invariant tori on which all Lyapunov exponents are zero. While KAM gives no information on the dynamics on the complement of the tori, it is believed that the Lyapunov exponents are "typically" non-zero, and thus regular and chaotic behaviour each occupy a region of phase space with positive volume.

More general examples of the KAM phenomenon (which do not arise as small perturbations of integrable Hamiltonian systems) have been given by Cheng and Sun [CS90], Herman [Her83], and Xia [Xia92] (see also [Yoc92]). In particular, if M is a manifold of dimension at least 2 and μ is a smooth measure on M, for sufficiently large r, $\mathrm{Diff}^r(M, \mu)$ contains an open set $\mathcal{U}$ such that for every $f \in \mathcal{U}$, the following hold:

(1) There is a family of codimension one invariant tori.
(2) The union of these tori has positive measure.
(3) On each torus, f is C^1 conjugate to a Diophantine translation; in fact, *all* Lyapunov exponents are zero on the invariant tori.

As with KAM theory itself, it is still not known exactly what happens on the complement of the invariant tori. A more completely understood class of examples are the *mushroom billiards* introducted by Bunimovich [Bun01], which have both regular and chaotic regions of phase space with positive volume. However, these maps are non-smooth; ideally one would like smooth examples. One such example has been recently constructed in [HPT10].

The situation is slightly better if instead of dealing with a single dynamical system one considers a one-parameter family of diffeomorphisms. To illustrate this approach we consider the following example.

Example 6.1. Przytycki [Prz82] and Liverani [Liv04] studied the following one-parameter family of area preserving diffeomorphisms of $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, which can be thought of as a perturbation of an Anosov diffeomorphism:

$$f_a(x, y) = (2x + y, x + y) + h_a(x)(1, 1) \pmod{2\pi}, \tag{6.1}$$

where $h_a(x) = -(1 + a)\sin x$. As the parameter a varies from $a = -1$ (unperturbed) to $a > 0$, the behaviour of the map changes:

(1) for $-1 \le a < 0$, f_a is Anosov;
(2) for $a = 0$, it has non-zero Lyapunov exponents almost everywhere;
(3) for $0 < a < \varepsilon$, it has an elliptic island (a regular region of phase space) and non-zero exponents almost everywhere outside of this island.

In particular, one can see that the map f_0 lies on the boundary of the set of Anosov diffeomorphisms; thus this example demonstrates a route from uniform hyperbolicity to non-uniform hyperbolicity and then to coexistence of regular and chaotic behavior. We remark that the Katok map also lies on the boundary of the set of Anosov diffeomorphisms.

This result exemplifies another approach to the problem of genericity: instead of considering open sets in $\mathrm{Diff}^{1+\alpha}(M,\mu)$, one may consider one-parameter families of diffeomorphisms and ask whether ergodicity and non-uniform hyperbolicity occur for a set of parameters of positive Lebesgue measure. One of the first successes of this approach was the seminal result by Jakobson [Jak81] on one-parameter families of unimodal maps, showing that such maps have absolutely continuous invariant measures (and hence stochastic behaviour) for a set of parameters with positive Lebesgue measure. There are multi-dimensional examples of families of volume (area) preserving maps where this is expected to occur, but for which the problem remains open. The most famous among them is the *standard (Chirikov–Taylor)* family of maps of the 2-torus [Chir79, Ras90]: these are given by $T_a(x,y) = (x', y')$ where

$$x' = x + a\sin(2\pi y) \pmod 1,$$
$$y' = y + x' \pmod 1.$$

Another interesting and quite promising example is a family of automorphisms of real $K3$ surfaces introduced by McMullen (see [MM02]).

7. From conservative systems to dissipative systems

When we attempt to move beyond conservative systems and treat dissipative systems, we are immediately confronted by the fact that there is no obvious “preferred” invariant measure. Thus we begin our analysis with a topological structure, not a measure-theoretic one, and consider an attractor $\Lambda = \bigcap_{n\geq 0} f^n(U)$, where U is a trapping region.

When the attractor Λ is hyperbolic, the “preferred” invariant measure is an SRB measure, which is a natural analogue of a smooth invariant measure for conservative systems. An invariant measure μ is an SRB measure if it has the following two properties:

(I) μ is hyperbolic (f has non-zero Lyapunov exponents μ-a.e.);
(II) for any measurable partition of a neighbourhood of M into local unstable leaves as in (2.11), the conditional measures of μ on the partition elements (local unstable leaves) are absolutely continuous with respect to the leaf volume (and hence to the conditional measures of Leb).

As described in Section 3 above, an SRB measure μ on a uniformly hyperbolic attractor Λ has the following “physicality” properties:

1. it is a weak* limit of the sequence of measures μ_n obtained from iterating a smooth measure under the dynamics of f (see (3.2));
2. the basin of attraction $B(\mu)$ (see (3.1)) has full volume in the trapping region U.

Now we are faced with the question: how do we generalise this to the case where the attractor fails to be uniformly hyperbolic, and is only non-uniformly hyperbolic? In this setting, it is not even immediately clear what the right question to ask is. To begin with, under what circumstances should we say that the attractor Λ is non-uniformly hyperbolic? The definition of non-uniform hyperbolicity requires a measure to give us a notion of "almost everywhere", and the whole heart of the problem just now is that we have no natural measure to work with on the attractor!

Indeed, one might consider the case in which there is a hyperbolic measure supported on Λ; however, there is no guarantee that this measure has anything to do with Lebesgue measure. In particular, it may not satisfy either of the physicality properties above; this is already true on hyperbolic attractors, where *any* invariant measure supported on Λ is hyperbolic, whether or not it is an SRB measure. Thus the absolute continuity property (II) is key for deriving the two "physicality" properties.

In light of this, we may ask what happens if Λ supports an SRB measure μ—that is, a hyperbolic measure (I) with absolutely continuous conditional measures on local unstable manifolds (II)—disregarding for the moment the question of whether or not such a measure actually *exists* in any given case.

Using absolute continuity of local stable manifolds, one can indeed show that the basin of attraction $B(\mu)$ has positive Lebesgue measure, recovering one of the physicality properties above. However, since the sizes of local stable manifolds may vary (and may be arbitrarily small), we cannot conclude that $B(\mu)$ has full measure in the trapping region U.

What we can say is that $A = \operatorname{supp} \mu$ is a *Milnor attractor* [Mil85]—that is, a closed subset of M such that

(1) the *realm of attraction* $\rho(A) = \{x \in M \mid \omega(x) \subset A\}$ has positive measure (recall that $\omega(x)$ is the set of accumulation points of the forward orbit of x);
(2) there is no strictly smaller closed set $A' \subset A$ such that $\rho(A')$ coincides with $\rho(A)$ up to a set of (Lebesgue) measure zero.

At this point there are still several important questions that remain unanswered. Here are two:

(1) Given a topological attractor Λ, when can we construct a Milnor attractor $A \subset \Lambda$ by the above method? That is, how do we obtain a non-uniformly hyperbolic SRB measure on Λ?
(2) Suppose a non-uniformly hyperbolic SRB measure supported on Λ exists. Under what conditions does the basin of attraction $B(\mu)$ have full measure in the trapping region U?

A natural approach to the first question is to *begin* with the first physicality condition above, and to consider a weak* accumulation measure of the sequence of measures μ_n from (3.2). Any such measure is supported on the topological attractor Λ, and one may wonder under what conditions μ has nonzero Lyapunov exponents. To this end, consider the following requirement on the map f:

$$\text{(H)} \qquad \chi^+(x, v) \neq 0 \text{ for Leb-a.e. } x \in U \text{ and every } v \in T_x M,$$

where U is a neighbourhood of the attractor Λ.

Conjecture 7.1. *There exists a diffeomorphism $f\colon M \to M$ and a trapping region U such that* (H) *holds but any accumulation measure of the sequence of measures μ_n has zero Lyapunov exponents almost everywhere in Λ.*

We believe that this conjecture is true, so that "poorly behaved" examples exist in which the natural construction of SRB measures fails. If this is so, then one has the following problem of great interest:

Problem 7.2. *What conditions should be added to the system so that* (H) does *imply that a physical measure (an accumulation measure of the sequence μ_n) is hyperbolic? is an SRB measure?*

For example, Alves, Bonatti, and Viana [ABV00] construct SRB measures in a similar manner under the assumption that the tangent bundle admits a dominated splitting into a uniformly contracting stable direction and a non-uniformly expanding centre-unstable direction. In this setting, one still has some of the geometric features of uniform hyperbolicity that are not present in the fully non-uniform case.

The following conjecture is in some sense the "mirror image" of Conjecture 7.1, and posits once again that the Lyapunov exponents of the limiting "physical" measure need not have anything to do with the Lyapunov exponents of Lebesgue typical points.

Conjecture 7.3. *There exists a diffeomorphism $f\colon M \to M$ and a trapping region U such that any accumulation measure of the sequence of measures μ_n has non-zero Lyapunov exponents almost everywhere in Λ, but* (H) *fails in the strongest possible sense: that is, for* Leb-*a.e.* $x \in U$, *we have* $\chi^+(x, v) = 0$ *for* **every** $v \in T_xM$.

8. Examples of hyperbolic SRB measures

There are a number of cases in which non-uniformly hyperbolic SRB measures are known to exist.

8.1. **The Hénon attractor.** The Hénon map is given by $f_{a,b}(x, y) = (1 - y - ax^2, bx)$, where a and b are real-valued parameters. In the seminal paper [BC91], Benedicks and Carleson, treating the map $f_{a,b}(x, y)$ as a small perturbation of the one-dimensional map $g_a(x) = 1 - ax^2$, developed a sophisticated techniques to describe the dynamics near the attractor. Building on this analysis, Benedicks and Young [BY93] established the existence of SRB measures for the Hénon attractor for certain parameter values.

Theorem 8.1. *There exist $\varepsilon > 0$ and $b_0 > 0$ such that for every $0 < b \le b_0$ one can find a set $A_b \in (2 - \varepsilon, 2)$ of positive Lebesgue measure with the property that for each $a \in A_b$ the map $f_{a,b}$ admits a unique SRB-measure.*

Wang and Young considered more general *rank one attractors*—that is, attractors for maps with one-dimensional unstable direction (corresponding to a one-dimensional map such as g_a) and the rest strongly stable (corresponding to a small perturbation of that map) [WY08].

In another direction, Mora and Viana [MV93] modified Benedicks and Carleson's approach in a way which allowed them to treat Hénon-like maps using some techniques from general bifurcation theory, such as homoclinic tangencies. Later, Viana [V93] extended results from [MV93] to higher dimensions; see [LV03].

8.2. **Partially hyperbolic attractors with negative (positive) central exponents.** Let Λ be a partially hyperbolic attractor for a C^2 diffeomorphism f. An invariant Borel probability measure μ on Λ is said to be a *u-measure* if the conditional measures $\mu^u(x)$ generated by μ on local unstable leaves $V^u(x)$ are absolutely continuous with respect to the leaf-volume on $V^u(x)$.

One can regard u-measures as the natural extension of SRB measures to the case of partially hyperbolic attractors. Indeed, it is shown in [PS82] that any accumulation measure of the sequence of measures μ_n in (3.2) is an f-invariant u-measure on Λ. Furthermore, any measure whose basin of attraction has positive volume is a u-measure (see [BDV05]), and if there is a unique u-measure for f on a partially hyperbolic attractor Λ, then its basin has full measure in the topological basin of Λ (see [Dol04]).

For hyperbolic attractors, topological transitivity of $f|\Lambda$ guarantees that there is a unique u-measure. In the partially hyperbolic situation, however, even topological mixing is not enough to guarantee uniqueness of u-measures. Consider $F = f_1 \times f_2$, where f_1 is a topologically transitive Anosov diffeomorphism and f_2 a diffeomorphism which is close to the identity. Then any measure $\mu = \mu_1 \times \mu_2$ (μ_1 is the unique SRB-measure for f_1 and μ_2 any f_2-invariant measure) is a u-measure for F. Thus, F has a unique u-measure if and only if f_2 is uniquely ergodic. On the other hand, F is topologically mixing if and only if f_2 is topologically mixing.

Let μ be a u-measure. Recall that f has *negative central exponents* if there is a subset $A \subset \Lambda$ with $\mu(A) > 0$ such that $\chi^+(x, v) < 0$ for any $x \in A$ and $v \in E^c(x)$. There are partially hyperbolic attractors for which *any* u-measure has zero central exponents (an example is the direct product of an Anosov map and the identity map in (4.1)); however, if f happens to have a u-measure with negative central exponents, then we can recover some of the general properties we are after.

Theorem 8.2 ([BDPP08])**.** *Assume that there exists a u-measure μ for f with negative central exponents and that for every $x \in \Lambda$ the global unstable manifold $W^u(x)$ is dense in Λ. Then*

(1) μ is the only u-measure for f and hence, the unique SRB measure;
(2) f has negative central exponents at μ-a.e. $x \in \Lambda$; f is ergodic and indeed, is Bernoulli;
(3) the basin of μ has full volume in the topological basin of Λ.

Observe that there are partially hyperbolic attractors which allow u-measures with negative central exponents, but for which not every global manifold $W^u(x)$ is dense in the attractor (for example, the product of an Anosov map and the map of the circle leaving north and south poles fixed).

The situation described in Theorem 8.2 is robust (stable under small perturbation of the map). More precisely, any C^1 diffeomorphism g sufficiently close to f in the C^1 topology has a hyperbolic attractor Λ_g which lies in a small neighborhood of Λ_f.

Theorem 8.3 ([BDPP08])**.** *Let f be a C^2 diffeomorphism with a partially hyperbolic attractor Λ_f. Assume that:*

(1) there is a u-measure μ for f with negative central exponents on a subset $A \subset \Lambda_f$ of positive measure;
(2) for every $x \in \Lambda_f$ the global strongly unstable manifold $W^u(x)$ is dense in Λ_f.

Then any C^2 diffeomorphism g sufficiently close to f in the $C^{1+\alpha}$-topology (for some $\alpha > 0$) has negative central exponents on a set of positive measure with respect to a u-measure μ_g. This measure is the unique u-measure (and SRB measure) for g, $g|\Lambda_g$ is ergodic with respect to μ_g (indeed is Bernoulli), and the basin $B(\mu_g)$ has full volume in the topological basin of Λ_g.

We now discuss the case of u-measures with positive central exponents. Alves, Bonatti, and Viana [ABV00] obtained an analogue of Theorem 8.2 under the stronger assumption that there is a set of positive volume in a neighbourhood of the attractor with positive central exponents.

Vasquez [V09] proved a result similar to Theorem 8.3 in the case of positive central exponents.

Theorem 8.4. *Let f be a C^2 diffeomorphism with a partially hyperbolic attractor Λ_f. Assume that:*

(1) there is a **unique** *u-measure μ for f with positive central exponents on a subset $A \subset \Lambda_f$ of* **full** *measure;*
(2) for every $x \in \Lambda_f$ the global strongly unstable manifold $W^u(x)$ is dense in Λ_f.

Then f is stably ergodic, i.e., all the conclusions of Theorem 8.3 hold.

The last two theorems motivate the following open problem (compare to Problem 4.6).

Problem 8.5. *Let f be a C^2 diffeomorphism with a partially hyperbolic attractor Λ and μ a u-measure with nonzero central exponents (of which some are positive and some are negative) on a set of positive measure. Assume that for every $x \in \Lambda$ the global strongly unstable manifold $W^u(x)$ is dense in Λ. Is μ ergodic? Is it a unique SRB measure for f? Is f stably ergodic?*

One may also attempt to understand what happens at another extreme by considering the case where *all* Lyapunov exponents vanish at Lebesgue-a.e. point, but the attractor itself (which has zero Lebesgue measure) is still partially hyperbolic. Compare the following with Conjecture 7.3:

Conjecture 8.6. *There exists a diffeomorphism $f\colon M \to M$ and a trapping region U such that*

(1) the attractor Λ is partially hyperbolic;
(2) (H) *fails in the strongest possible sense (for* Leb-*a.e. $x \in U$, we have $\chi^+(x,v) = 0$ for every $v \in T_xM$;*
(3) any accumulation measure of the sequence of measures μ_n obtained from iterating a smooth measure under the dynamics of f is a u-measure.

8.3. **Generalized hyperbolic attractors.** Let M be a smooth Riemannian manifold, $N \subset M$ a closed subset, and $f\colon M \setminus N \to M$ a C^2 diffeomorphism. N is called the *singularity set*, and we assume the following behaviour of f near N: there exist constants $C > 0$ and $\alpha \geq 0$ such that

$$\|d_x f\| \leq C\rho(x,N)^{-\alpha}, \quad \|d_x^2 f\| \leq C\rho(x,N)^{-\alpha} \quad x \in M \setminus N.$$

(Here $\rho(x,N)$ denotes the distance from x to N.) If U is a trapping region we set

$$U^+ = \{x \in U : f^n(x) \notin N \text{ for all } n \geq 0\}$$

and define the *generalized attractor* for f by

$$\Lambda = \bigcap_{n\geq 0} f^n(U^+)$$

The set Λ is invariant under both f and f^{-1}.

Let us fix $\varepsilon > 0$ and set for $\ell \geq 1$,

$$\Lambda^{\pm}_{\varepsilon,\ell} = \{z \in U \mid \rho(f^{\pm n}(z), N) \geq \ell^{-1}e^{-\varepsilon n} \text{ for all } n \geq 0\},$$
$$\Lambda^{\pm}_{\varepsilon} = \bigcup_{\ell\geq 1} \Lambda^{\pm}_{\varepsilon,\ell}, \quad \Lambda^0_\varepsilon = \Lambda^+_\varepsilon \cap \Lambda^-_\varepsilon.$$

The set Λ^0_ε is f- and f^{-1}-invariant and $\Lambda^0_\varepsilon \subset \Lambda$ for every ε. This set is an "essential part" of the attractor and in general, may be empty. The following condition guarantees that Λ^0_ε is not empty: there exist $C > 0$, $q > 0$ such that for any $\varepsilon > 0$ and $n \geq 0$,

$$\text{Leb}(f^{-n}(B(\varepsilon,N) \cap f^n(U^+))) \leq C\varepsilon^q, \tag{8.1}$$

where $B(\varepsilon, N) = \{x \mid \rho(x,N) < \varepsilon\}$. If the map f is uniformly hyperbolic on U, then the attractor Λ is called a *generalized hyperbolic attractor*, and it is indeed a non-uniformly hyperbolic set for f. Examples of generalized hyperbolic attractors include the geometric Lorenz attractor, the Lozi attractor, and the Belych attractor. It is shown in [Pes92] that any accumulation point of the sequence of measures (3.2) is an SRB measure for f.

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