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Geodesic flows with hyperbolic behaviour of the trajectories and objects connected with them

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Contents

Introduction

1. The history of the investigation of geodesic flows is briefly presented in Anosov's introduction to the translation of the book [21] by Klingenberg. The direction this survey takes goes back to Hadamard. From the first papers onwards, differential-geometric methods have played a large, and at times even a predominant, part in the investigation. Nevertheless, already then an exceptionally important observation was made, namely, that the topological and ergodic properties of a flow, when the curvature is negative, are in essence determined by instability of the trajectories. However, there were not (and evidently there could not have been) any exact conceptions on either of what should be understood by this instability nor, relying only on it, of how to deduce topological and ergodic properties of the flow. The results obtained in this period were reflected in the survey [71] by Hedlund (see also the article [44] by Hopf), which was written in the late 30's and summarized, as it were, the whole preceding research.

In the subsequent years the efforts of mathematicians were directed to the study of geodesic flows on manifolds of negative curvature. Up to 1962 only isolated and partial results were obtained, until the papers [1], [2], [37], and [40] were published, and a little later in 1967 the familiar book by Anosov [3][(1)] and the article by Anosov and Sinai [4], which include detailed proofs of the results. These papers not only increased our knowledge of geodesic flows significantly, but also constituted a move away from their investigation by differential-geometric methods towards dynamical techniques. Nowadays geodesic flows are regarded as a particular (though important) case of a special class of dynamical systems, the so-called Y-systems. The instability of trajectories mentioned above has finally been characterized exactly by means of the so-called uniform conditions of hyperbolicity,[(2)] and forms the basis for an axiomatic definition of these systems. In the works by Anosov and Sinai cited above there is a fairly full description of the ergodic and topological properties of Y-systems, and it is established that geodesic flows on compact manifolds of negative curvature are Y-flows. It now remained only to reap the fruits by applying the results of the general theory. This application was almost automatic—there were no further difficulties of a geometric nature. Here are only a few of the results: geodesic flows are topologically transitive (and almost every trajectory is everywhere dense) and even mix topologically; they have the strongest possible ergodic properties (and, if we include the latest work, even the Bernoulli property). The methods of the theory of dynamical systems have enriched differential geometry too: important results have been obtained about compact Riemannian manifolds of negative curvature. For example, it turned out that closed geodesics are everywhere dense, and an exact and simple formula for the asymptotic behaviour of closed geodesics has been found (Margulis [26]), horospheres have been constructed and it has been proved that each of these is everywhere dense; also that almost every geodesic is dense in the manifold and so on. This was a triumph of the theory of Y-systems. The matter is not confined to manifolds of negative curvature: the boundaries have been widened.

The study of manifolds of negative curvature is connected in part with the question (important for Riemannian geometry) of the influence of the curvature on the properties of a manifold. Research in this direction, especially that mentioned above, showed however that in many cases this influence is governed not only by curvature, but also by certain properties of geodesic flow. For we can take a metric in which a geodesic flow is a Y-flow, but which has certain parts of positive curvature (such metrics exist and may even comprise an open set in the space of metrics!; see for

(1) A short summary of the earlier results is included in §3 of the book.

(2) Roughly speaking, these conditions mean that the picture of the behaviour of trajectories in a given neighbourhood is like that of trajectories near a saddle point; an exact definition is given in §1.9.

example, [70]). Again, the results of the general theory of Y-systems allow us to study the topological and ergodic properties of a geodesic flow, and also to obtain interesting information both about the Riemannian metric itself (for example, it turns out that it does not have conjugate points) and about the global geometry of the manifold (Klingenberg [82]). Such manifolds are said to be of Anosov type. The study of them led naturally to an analysis of the question when a geodesic flow is a Y-flow. Eberlein [53], [54], obtained interesting results in this direction. Closely related to manifolds of Anosov type are manifolds of hyperbolic type, that is, those that admit a metric of negative curvature.(1) The latter arose even earlier in differential geometry: and in any case, Morse [94] already understood that on surfaces with negative Euler characteristic (which admit a metric of constant negative curvature) the geodesic flow, as it were, to some extent inherits properties of flow in a metric of negative curvature (see also [104]). In this connection we point to an interesting but so far unsolved question: is a manifold of Anosov type of hyperbolic type? (see [56], [84]).

However, the penetration of dynamical methods into differential geometry does not stop here. There arises the interesting and timely problem as to whether it is possible to extend the class of Riemannian metrics so that a geodesic flow still has certain hyperbolic properties (though possibly weaker). The natural candidate is the class of Riemannian metrics without focal points (and in many cases the more general class of Riemannian metrics without conjugate points).(2) Episodic results on this topic are of an even earlier date ([65], [95]). More recently, investigations in this direction have stimulated the study of the ergodic properties of dynamic systems with non-zero characteristic Lyapunov indicators (see [32], [33]). One can say with complete certainty that geodesic flows have already served as a source (and as an excellent model) for the development of methods in the theory of dynamical systems of hyperbolic type. But as soon as such a theory was created the geodesic flows themselves became one of the most important domains for its application.

On this very path there arose difficulties of a purely geometric nature. Fortunately, at the same time there appeared a group of remarkable papers by Eberlein (see [50]–[54]) in which these geometric problems were

(1)We recall once more that in this metric a geodesic flow is a Y-flow.

(2)We mention that the conditions for the absence of focal and even of conjugate points (along all geodesics!) are more complex than the conditions for negative curvature, in the following respects. Firstly, they are difficult to check since they are global (in contrast to the conditions for negative curvature, which are local), although manifolds without conjugate and without focal points constitute an important class for Riemannian geometry, and their geometric properties have been widely studied (in [54] and [70], for example, there are geometric criteria for the absence of focal points). Secondly, generally speaking, they are not coarse (so that the corresponding metrics, in contrast to the case of negative curvature, do not, generally speaking, form an open set in the space of the metrics).

either solved or provided with an apparatus for their solution. Thus, for Riemannian metrics without conjugate points there were studied limiting solutions of the Jacobi equation, invariant distributions were constructed that laid claim to the role of stable and unstable subspaces in the hyperbolic theory, and so on. Later, in other works by Eberlein, by the present author, and by others (see [52], [106], [34], [35], [60], [75]) it was proved that these distributions are integrable and that their maximal integral manifolds form invariant layers in the geodesic flow. The projections of these integral manifolds form limiting spheres—objects that undoubtedly play an important part in differential geometry. All this made it possible to apply the results achieved for systems with non-zero Lyapunov indicator and to study the ergodic properties of geodesic flows on manifolds without focal points (or even without conjugate points).

The works by Eberlein and other authors mentioned above brought about a turn in the investigations of geodesic flows from dynamical methods to the methods of differential geometry, by setting up a balance between them. Thus, relying on arguments of a purely geometric nature and quite apart from the results given above, it became possible to gain fairly full and thorough information of a topological nature (which had not been possible within the framework of the general theory of systems with non-zero Lyapunov indicator). It turned out that on a compact manifold without conjugate points (under certain additional assumptions), a geodesic flow is transitive, the limiting spheres are everywhere dense on the manifold (this has been proved, for example, for dimension 2), and so on.

2. Side-by-side with geodesic flows there occur in Riemannian geometry other dynamical systems closely connected with them, the so-called horocycle flows and frame flows. Horocycles have always played an important part in non-Euclidean geometry from its very beginnings, but an exact definition of a horocycle flow in the case of a surface of negative curvature was given by Hopf and Hedlund in the 30's. The study of frame flows is of comparatively recent date: a general definition of them was given in 1961 by Arnol'd [5].

The study of frame flows on a manifold of negative curvature came about just when the theory of dynamical systems of hyperbolic type was developed. Obviously, this was no accident—here, as in a mirror, is reflected the history of geodesic flows. Again, geometry was one of the sources(1) of the development of hyperbolic methods leading to the formation of a theory for the so-called partial hyperbolic dynamical systems (see [12]), and the results of an ergodic and topological character established within the framework of this theory led to a study of the corresponding properties for frame flow

(1)Other sources were investigations of an algebraic nature (see [42], [101]) and also the natural question arising within the framework of the hyperbolic theory, of weakening the requirements of hyperbolicity in the direction of a transition from "full" to "partial" hyperbolicity.

(see [12], [10], [11]). Subsequent investigations, in turn, required the application of new geometric concepts (see [47]).

The theory of horocycle flows developed similarly. In the case of negative curvature it was possible to study ergodic and topological properties relying on theorems from the general theory of Y-systems; the most complete results here were obtained by Marcus [91]-[92] (see also [49]). In extending these results to wider classes of metrics (without focal and without conjugate points) it was necessary for the investigation of ergodic properties to turn to the theory of systems with non-zero characteristic Lyapunov indicators; and for investigating topological properties supplementary arguments of a geometric character were used (Eberlein [55]).

3. Mention has been made above of the role played by geodesic flows in the study of the geometry of manifolds. But their importance grows if we take into account the fact that geodesic flows are phase flows of certain conservative mechanical systems; even in the first papers on geodesic flows they were regarded as models for isolated problems in classical and celestial mechanics. There are great possibilities in this direction, but we confine ourselves to a survey of works since 1967 on geodesic flows with hyperbolic behaviour of the trajectories (in the sense in which we spoke of this earlier). We are deliberately leaving aside other types of geodesic flows (for example, we do not look at integrable Hamiltonian systems). Moreover, we consider only compact manifolds. This is stipulated not only by the rigid constraints on a journal paper (for the investigation of non-compact manifolds would increase the size of the paper considerably), but also because in the compact case the results are most complete. Of course, many of the theorems to be proved later remain true in the non-compact case (in as much as the corresponding arguments use not so much compactness as conditions of uniformity); however, on the whole, the latter case is richer and more complicated than the compact one(1) (in a few places in the survey we mention this again). Also, we do not touch on geodesics on Finsler manifolds.

I should like to express my warmest thanks to D.V. Anosov with whom I frequently consulted during the work on the survey. He read the manuscript and made many useful remarks and additions. He also was kind enough to draw my attention to a number of works (and even to put them at my disposal) that I did not know and which are now duly included in the survey.

(1)We mention that in the papers of Morse, Hopf, and Hedlund in the 30's (see [44], [94], [77], [73], [74]) there were considered also non-closed surfaces of negative curvature, and definitive results of a topological and ergodic character were obtained (topological transitivity, ergodicity, and mixing; the latter under the assumption of finite volume). However, their generalization to the many-dimensional case or to Riemannian metrics without conjugate points comes up against considerable difficulties and is not yet final (especially with regard to ergodic results; certain results of a topological nature have been obtained by Eberlein [58]).

PART I

GEODESIC FLOWS

§1. Preliminary information from differential geometry, topology, and ergodic theory

1. In the present survey we use diverse material from differential geometry, topology, the theory of smooth dynamical systems, and ergodic theory. For the convenience of the reader we collect here some of the more important concepts that occur frequently; the selection is naturally subjective and by no means complete. For more detailed information we can recommend the following monographs: on differential geometry [9], [17], [21], [29]; on differential topology [19], [25], [43]; on ergodic theory [8], [39], [22]; on dynamical systems [6], [16], [30]. Since we have to deal primarily with the methods of the hyperbolic theory, we can recommend the surveys [20] and [36], where there is a fairly full bibliography.

2. **Riemannian metrics** (see [9], Ch. 7 and 8; [17], §§3 and 5; [29], Ch. II). By a manifold M we mean in this work a smooth (of class C^∞) P-dimensional compact manifold without boundary, endowed with a Riemannian metric of class C^∞ denoted by $\langle\, ,\, \rangle$ (special cases when the manifold is not compact or the Riemannian metric has finite smoothness will be pointed out separately). We denote by TM the $2p$-dimensional tangent bundle to M and by π: $TM \to M$ the natural projection.

The presence of a Riemannian metric allows us to introduce on M various objects and structures compatible with it. We mention them only in introducing the corresponding notation, and we dwell only on some of them in more detail.

1) The norm $\|\cdot\|$ in each tangent space T_xM, $x \in M$.

2) The *distance* ρ between two points x and y (defined as the lower bound of lengths of possible curves in M joining x and y).

3) The *Riemannian connectivity* (Levi-Civita connectivity) and the *covariant product* induced by it (denoted by "$'$" or $\frac{\partial}{\partial t}$) and the *curvature tensor* $R(X, Y)Z$ (where X, Y, and Z are smooth vector fields on M or along a curve).

4) The *connectivity map* K: $TTM \to TM$ defined by $K\xi = Z'(0)$, where $\xi \in TTM$ and $Z(t)$ is any curve in TM tangential to ξ when $t = 0$. The map K is linear and continuous, and its kernel forms a p-dimensional *space of vertical vectors* V_v. The *space of horizontal vectors* V_h is defined as the p-dimensional kernel of the linear operator $d\pi$. For each $v \in TM$ there exists the representation

$$T_vTM = (V_v)_v \oplus (V_h)_v,$$

which allows us to introduce in TM a *canonical Riemannian metric* by

$$\langle \xi, \eta \rangle = \langle K\xi, K\eta \rangle + \langle d\pi\xi, d\pi\eta \rangle, \quad \xi, \eta \in TTM,$$

in which the spaces V_v and V_h are orthogonal.

5) *Geodesics*—these are curves along which the tangential vector field is parallel. For any $x \in M$ there exists a local chart in which for any $v \in T_xM$ the geodesic $\gamma_v(t) = (\gamma_1(t), \ldots, \gamma_p(t))$ with $\gamma_v(0) = x$, $\dot{\gamma}_v(0) = v$ satisfies a certain system of second-order differential equations. We always suppose that for the geodesic $\gamma(t)$ the parameter t is chosen as the arc length; thus, $\|\dot{\gamma}(t)\| = 1$ and $(\dot{\gamma}(t))' = 0$. On complete (in particular, on compact) manifolds geodesics are infinitely extendable, so that t takes all values from $-\infty$ to ∞. For any points $x, y \in M$ there exists a geodesic joining x and y (generally speaking, non-unique). Among such geodesics there always is one whose length is equal to $\rho(x, y)$. For any $v \in TTM$ there exists a unique[(1)] geodesic, denoted by $\gamma_v(t)$, for which $\gamma_v(0) = \pi(v)$, $\dot{\gamma}_v(0) = v$.

6) The *exponential map* $\exp_x$: $T_xM \to M$ associates a point $\gamma_v(1)$ with a vector $v \in T_xM$. For complete manifolds $\exp_x$ is defined for all $v \in T_xM$, and is surjective. It is diffeomorphic in the sphere $B(r_x)$ in T_xM with radius r_x (the radius of injectivity of the exponential map). In the case of compact manifolds $r = \min_{x \in M} r_x > 0$.

7) The *Lebesgue dimension* ν, defined on the $\mathfrak{S}$-algebra of all Borel subsets of M (one of the methods of constructing it is given in §2.2).

If $N \subset M$ is a smooth submanifold, then the restriction of the Riemannian metric $\langle\ ,\ \rangle \mid N$ gives a Riemannian metric on N and, consequently, induces objects analogous to those above. In the appropriate notation we indicate the subset symbol by a lower index.

3. The universal Riemannian covering manifold.

Let M be a connected Riemannian manifold. Its *universal covering Riemannian manifold* is denoted by H (the construction of H is described in, for example, [17], §7.2). The fundamental group $\pi_1(M)$ acts by *isometries* on H and is a proper discrete subgroup of the full group of isometries; here $M = H/\pi_1(M)$. Conversely, if Γ is such a subgroup, then it is the fundamental

(1) There is another (more topological rather than differential) approach to the definition of geodesics as curves of which sufficiently small sections have the smallest length among all curves joining two given points (and at the same time attaining a distance between two nearby points). This definition mades sense in an arbitrary metric space; however, in the general case, geodesic need not exist or need not have the "usual" properties. But if the metric spaces satisfy a number of axioms of a geometric nature, then geodesic exist and have the "usual" properties, although they are not, generally speaking, smooth curves. This approach is developed in the interesting book [13] by Busemann, and he calls the corresponding metric spaces G-spaces (nowadays, this term is used in a different sense, so that it is better to call them Busemann spaces).

By subjecting a Busemann space to various axioms, we can obtain analogues (and generalizations) of classical projective and affine geometry, of the geometries of Minkowsky and Hilbert, of hyperbolic geometry, and so on. On the whole this approach is more in the spirit of traditional geometry than the differential methods. However, to obtain for example, ergodic results it is necessary to postulate a definite smoothness of the manifold and the metric, sometimes even greater than that which guarantees the existence of smooth geodesics. Naturally, differential methods are more adequate here.

group on the manifold $M = H/\Gamma$. If M is compact for some subgroup Γ, then we say that H admits a *compact factor.*

The covering map $H \to M$ allows us to "lift" objects we shall later construct for M to the manifold H (and conversely to "lower" objects on H into M. This procedure does not, as a rule, present any difficulty, and we do not consider it in detail on every occassion. Corresponding objects in H and M are denoted by the same letters.

4. **Curvature** (see [9], Ch. 5, 6, and 9; [17], §§2 and 3).
For any $x \in M$ and any two-dimensional space $P \subset T_xM$ given by vectors $v, w \in T_xM$ there is defined the quantity

$$K_x(P) = \frac{\langle R(v, w)w, v\rangle}{\|v\|^2\|w\|^2 - \langle v, w\rangle^2},$$

which is called the *curvature* of M at x in the two-dimensional direction P (it can be shown that $K_x(P)$ does not depend on the choice of v and w specifying the plane P). The Riemannian metric is said to have *negative (non-positive)* curvature if the curvature at each point and in each two-dimensional direction is negative (non-positive).

5. **The Fermi co-ordinates** (see [17], §3.8).
We consider any geodesic $\gamma(t)$ and fix an arbitrary orthonormal basis $\{e_i(t)\}$ $(i = 1, 2, \ldots, p)$ in the space $T_{\gamma(0)}M$ such that $e_n = \dot\gamma(0)$. Let $e_i(t)$ be the vector field along $\gamma(t)$ obtained by a parallel shift of e_i along $\gamma(t)$. Then the vectors $\{e_i(t)\}$ $(i = 1, 2, \ldots, p)$ form an orthonormal basis in $T_{\gamma(t)}M$. The corresponding coordinates in $T_{\gamma(t)}M$ are called the Fermi coordinates. (We do not need to go to coordinates on M itself, as is usually done.)

6. **Jacobi fields** (see [9], §9.4; [17], §4.2; [29], Ch. II).
The Jacobi field along a geodesic $\gamma(t)$ is vector field $Y(t)$ satisfying the Jacobi equation

$$Y''(t) + R(X, Y)X = 0, \tag{1.1}$$

where $X(t) = \gamma(t)$. In the Fermi coordinates $\{e_i(t)\}$ $(i = 1, 2, \ldots, p)$ along $\gamma(t)$, (1.1) can be rewritten in the form of the second-order linear differential equation

$$\frac{d^2}{dt^2}Y(t) + R(t)Y(t) = 0, \tag{1.2}$$

where $Y(t) = (Y, (t), \ldots, Y_p(t))$ is a vector, and $R(t) = (K_{ij}(t))$ is a matrix whose element $K_{ij}(t)$ is the curvature at $\gamma(t)$ in the two-dimensional direction given by the vectors $e_i(t)$ and $e_j(t)$. From this it follows that the Jacobi field is defined for all t and is given by $Y(0)$ and $Y'(0)$. The space of Jacobi fields is $2p$-dimensional and is denoted by $J(\gamma)$.

A Jacobi field is said to be *orthogonal* to the velocity vector (or simply "orthogonal") if $\langle Y(t), \dot\gamma(t)\rangle = 0$ for all t. In this case $\langle Y'(t), \dot\gamma(t)\rangle = 0$ for each t. A Jacobi field is orthogonal if $\langle Y(0), \dot\gamma(0)\rangle = \langle Y'(0), \dot\gamma(0)\rangle = 0$. The space of orthogonal Jacobi fields is $(2p-2)$-dimensional, and is denoted by $J_0(\gamma)$.

Together with the Jacobi vector equation (1.2) we consider the *Jacobi matrix equation* of the form

$$\frac{d^2}{dt^2} D(t) + R(t) D(t) = 0, \tag{1.3}$$

where $D(t) = (d_{ij}(t))$, $(i, j = 1, \ldots, p)$ is a matrix. (1.3) describes the complete set of linearly independent solutions of (1.2). For any two solutions X and Y of (1.3) the "Wronskian"

$$W(X, Y)(t) = \frac{d}{dt} X^*(t) \circ Y(t) - X^*(t) \circ \frac{d}{dt} Y(t)$$

("*" denotes the transposed matrix) is constant. If a solution $D(t)$ of (1.3) is non-degenerate in an interval (t_0, t_1), (that is, $\det(D(t)) \neq 0$ for $t \in (t_0, t_1)$), then the matrix function $U(t) = \frac{d}{dt} D(t) \circ D^{-1}(t)$ is a solution on that interval of the Riccati matrix equation

$$\frac{d}{dt} U(t) + U(t)^2 + R(t) = 0, \tag{1.4}$$

where $U(t)$ is symmetrical if and only if $W(D, D)(t) = 0$ on (t_0, t_1). More will be said in §2, 3 and 4 about some other properties of the solutions of the Jacobi and Riccati equations.

The Jacobi equations arise in the calculus of variations (in the formula for the second variation of the geodesic). An account of these questions is not part of our task (the topic is treated in [17], §4 and [29], Ch. II, see also below §2.2). However, we quote one result showing how to describe the infinitesimal properties of geodesics by means of Jacobi fields. Let $\gamma(t)$ be a geodesic and $Z(s)$, $-\varepsilon \leqslant s \leqslant \varepsilon$ a curve in TM with $Z(0) = \dot\gamma(0)$. We look at the *variation* $r(t, s)$ of γ of the form

$$r(t, s) = \exp(tZ(s)), \quad t \geqslant 0, \quad -\varepsilon \leqslant s \leqslant \varepsilon. \tag{1.5}$$

Proposition 1.1 (see [29], Lemma 14.3). *The vector field*

$$Y(t) = \frac{\partial}{\partial s} r(t, s)\,|_{s=0} \tag{1.6}$$

along the geodesic γ is the Jacobi field along y.

7. **Conjugate points** (see [9], §11.3 and [17], §4.3).
Let $\gamma(t)$ be a given geodesic. Two points $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$ are said to be *conjugate* if there exists a Jacobi field $Y(t)$ along γ that does not vanish identically and is such that $Y(t_1) = 0$ and $Y(t_2) = 0$. A Riemannian metric (and, loosely speaking, the manifold itself) is said not to have conjugate points if no two points are conjugate along any geodesic in M. In what follows we consider only Riemannian metrics without conjugate points.

The absence of conjugate points along $\gamma(t)$ is equivalent to saying that for any t_1 and t_2 and for any vectors Y_1 and Y_2 there exists a unique Jacobi field $Y(t)$ such that $Y(t_1) = Y_1$ and $Y(t_2) = Y_2$ (thus, the corresponding boundary-valued problem is soluble, in fact uniquely, for any boundary conditions). We give a geometric criterion for the presence of conjugate points.

***Proposition* 1.2** (see [17]). *Two points $\gamma(0)$ and $\gamma(t_1)$, $t_1 > 0$, are conjugate if and only if there is a variation $r(t, s)$ of the form (1.5), with $r(0, s) = \gamma(0)$, $-\varepsilon \leqslant s \leqslant \varepsilon$, for which $\gamma(t_1)$ is a limiting point of intersection of the geodesics $\gamma(t)$, and $t \to r(t, s)$ as $s \to 0$ and t is close to t_1.*

Although manifolds without conjugate points may have "small" parts of positive curvature, nonetheless, they have many properties of manifolds of negative curvature.(1) To illustrate this we denote by $S^{p-1}(x, t)$ the $(p-1)$-dimensional sphere on the universal Riemannian covering manifold H.

***Proposition* 1.3.** *Suppose that a Riemannian metric on M does not have conjugate points. Then*

1) *H is diffeomorphic to $\mathbf{R}^n$ by means of the map $\exp_x$ for any $x \in H$;*

2) *a geodesic in H attains the distance between any two of its points;*

3) *any two geodesics in H intersect at most once; any two intersecting geodesics $\gamma_1(t)$ and $\gamma_2(t)$ in H diverge, that is,*

$$\rho(\gamma_1(t), \gamma_2(t)) \to \infty \quad as \quad t \to \infty;$$

4) *for any $x \in H$, $t > 0$, and $y \in S^{p-1}(x, t)$ the geodesic $\gamma_{xy}(s)$ is orthogonal to the sphere $S^{p-1}(x, t)$.*

1) is called the Hadamard-Cartan theorem (see [17], §7); 2) is proved, for example, in [17], §7.2; 3) for the two-dimensional case is proved by Green in [67] (see Theorem 3.1); in the general case by Eberlein in [50], 168. A generalization of this assertion to manifolds with *poles* is given in [62]. A proof of 4) can be found, for example, in [29], Lemma 10.5.

8. Focal points (see [9], §11.3 and [17], §4.3).
Let $\gamma(t)$ be a geodesic. Two points $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$ are said to be *focal* if there exists a Jacobi field $Y(t)$ along γ such that $Y(t_1) = 0$, $Y'(t_1) \neq 0$, and $\frac{d}{dt}(\| Y(t) \|^2)\big|_{t=t_2} = 0$. We say that a Riemannian metric (and, loosely speaking, the manifold itself) does not have focal points if no two points are focal along any geodesic in M. (In this case, as Goto has shown (see [64], Theorem 1), for any geodesic γ and any Jacobi field $Y(t)$ along γ for which $Y(0) = 0$ and $Y'(0) \neq 0$, the function $\varphi(t) = \| Y(t) \|$ tends monotonically to $+\infty$ as $t \to +\infty$. If a Riemannian metric does not

(1)We mention that when the curvature is negative there are no conjugate points (see also §1.8).

have focal points, then it does not have conjugate points. If a manifold has non-positive curvature, then it does not have focal points.

The absence of focal points means that $\|Y(t)\|$ is monotonic for any Jacobi field $Y(t)$. This circumstance (taking Proposition 1.1 into account) allows us to establish the following geometric properties of manifolds without focal points.

Proposition 1.4 (see [54]). *Suppose that a manifold M has no focal points. Then*

1) *for any two geodesics $\gamma_1(t)$ and $\gamma_2(t)$ on H the function $\varphi(t) = p(\gamma_1(t), \gamma_2(t))$, $t \in \mathbf{R}$, to convex;*

2) *the sphere $S^{p-1}(x, t)$ is a convex set.*

We introduce here two criteria, the first of which establishes the presence of focal points, and the second their absence.

Proposition 1.5. 1) *Two points $\gamma(t_1)$, $t_1 > 0$, and $\gamma(0)$ are focal if and only if there exists a variation $r(t, s)$ of the form* 1.5 *such that the curve $\pi(Z(s))$ is a geodesic and $\gamma(t_1)$ is a limiting point of intersection of the geodesics $\gamma(t)$ and $t \to r(t, s)$ as $s \to 0$ and t is close to t_1 (see* [17]).

2) *A Riemannian metric on M does not have focal points if and only if for any geodesic $\gamma \in H$ and any point $p \in H$ not lying on γ there exists a unique geodesic σ in H passing through p orthogonally to γ. Moreover, $\varphi(t) = \rho(p, \gamma(t))$ is differentiable, $\varphi' > 0$ and $\varphi'' = 0$ at the point $t = t_0$ only, and the geodesic joining p and $\gamma(t_0)$ is just perpendicular dropped from p to γ (for* $\dim M = 2$, *this was proved by Green in* [67]; *in the general case, by Eberlein in* [54], *Proposition* 2).

Remark 1.1. Propositions 1.1–1.5 are true not only for compact but for arbitrary complete Riemannian manifolds [note that in 3) of Proposition 1.3 it must be required, in addition, that either the curvatures $K_x(P)$ be uniformly bounded below (see [50], p.168) or that the metric does not have focal points (see [103], Proposition 2, and [64], Theorem 2]. O'Sullivan gained interesting information on the structure of the fundamental groups of compact manifolds without conjugate or focal points (see [102], [103], also [57], [106], [59]).

9. Flows with hyperbolic behaviour of the trajectories.
Let f^t: $M \to M$ be a *flow* on a manifold M of class C^r, $r \geqslant 1$, given by a *vector field X*. We denote by Z_x the one-dimensional subspace of T_xM generated by the vector $X(x)$, and by W^0 the smooth invariant one-dimensional *layer* generated by the trajectories of the flow.

The flow f^t gives rise to a flow F^t in the tangent bundle TM, acting according to the formula

$$(1.7) \qquad F^t(x, v) = (f^t(x), df^t_x v), \quad x \in M, \quad v \in T_xM.$$

The vector field specifying the flow F^t (and defined on TM) is called the *vector field of variation* (in local coordinates the classical "variational equations" correspond to it), and the trajectories of F^t are called solutions of the *variational equations*. These equations describe the infinitesimal properties of the trajectories of the flow f^t.

A flow f^t is called *Y-flow* if there exist two continuous distributions E^- and E^+ such that

1) $T_xM = E_x^- \oplus E_x^+ \oplus Z_x$ for each $x \in M$;
2) $df^tE_x^- = E_{f^t(x)}^-$, $df^tE_x^+ = E_{f^t(x)}^+$ for any $x \in M$ and $t \in \mathbf{R}$;
3) there exist constants $c < 0$ and $\lambda \in (0, 1)$ such that for any $x \in M$, $t > 0$,

$$(1.8)\qquad \begin{cases} \| df^tv \| \leqslant c\lambda^t \| v \| & \text{if} \quad v \in E_x^-, \\ \| df^tv \| \geqslant c^{-1}\lambda^{-t} \| v \| & \text{if} \quad v \in E_x^+. \end{cases}$$

The subspaces E_x^- and E_x^+ are said to be, respectively, *contracting* (*stable*) *and expanding* (*unstable*). The distribution E^- (or E^+) is *integrable and its maximal integral submanifolds form a continuous variable fibration*, denoted by W^- (or W^+). The fibration W^- (or W^+) is *contracting* (*or expanding*): there exists a $K > 0$ such that for any $x \in M$, $y \in W^-(x)$ (or $y \in W^+(x)$), and for any $t > 0$ (or $t < 0$)

$$(1.9)\qquad \rho^-(f^t(x), f^t(y)) \leqslant K\lambda^t\rho^-(x, y) \ (\text{or}\ \rho^+(f^t(x), f^t(y)) \leqslant K\lambda^t\rho^+(x, y)),$$

where ρ^- (or ρ^+) denotes the distance on the corresponding fibre of W^- (or W^+) induced by the Riemann metric (see § 1.1). The distributions $E^- \oplus Z$ and $E^+ \oplus Z$ are also integrable. The corresponding filtrations are denoted by W^{-0} and W^{+0}. Here $W^{-0}(x)$ is also called *the stable sheet* at x and is defined as follows:

$$W^{-0}(x) = \bigcup_t f^t(W^-(x)) = \bigcup_t W^-(f^t(x)).$$

$W^{+0}(x)$ is also called *the unstable sheet at* x and is defined analogously.

The *characteristic Lyapunov indicator* for the flow f^t is the function $\chi^+ : TM \to \mathbf{R}$ defined by

$$\chi^+(x, v) = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \| df^tv \|, \quad x \in M, \quad v \in T_xM.$$

We assume that f^t preserves a certain Borel measure μ. We say that f^t has *non-zero characteristic Lyapunov indicators* if the set

$$(1.10)\qquad \Lambda = \{x \in M : \chi^+(x, v) \neq 0 \quad \text{for every} \quad v \in T_xM \setminus Z_x\}$$

has positive measure. On Λ the flow f^t satisfies the conditions of non-uniform hyperbolicity, which is analogous to 1), 2), 3) with this difference only that in 3) the constants c and λ are replaced by certain measurable functions on M.

We do not dwell here in more detail on the theory of Y-flows with non-zero characteristic Lyapunov indicators, referring the reader à propos the first to [3] and [4], and à propos the second to [32], [33], and [80]. About the results of these works that we used in what follows we have more to say in the course of the exposition.

In conclusion we indicate another version of weakening of the requirements of hyperbolicity, which consists in the transition to so-called partial hyperbolicity. Mor exactly, a flow f^t on a manifold M is said to be partially hyperbolic (see [12]) if there exist distributions E^-, E^+, and E^0 such that

1) $T_xM = E_x^- \oplus E_x^+ \oplus E_x^0, \quad Z_x \subset E_x^0;$

2) $df^tE_x^- = E_{f^t(x)}^-, \quad df^tE_x^+ = E_{f^t(x)}^+, \quad df^tE_x^0 = E_{f^t(x)}^0$ for all $x \in M$;

3) there exist constants $c > 0$, $0 < \lambda_1 \leqslant \mu_1 < \lambda_2 \leqslant \mu_2 < \lambda_3 \leqslant \mu_3$, $\mu_1 < 1 < \lambda_3$, such that for all $t > 0$

$$c^{-1}\lambda_1^t \| v \| \leqslant \| df^t v \| \leqslant c\mu_1^t \| v \|, \quad v \in E_x^-,$$
$$c^{-1}\lambda_2^t \| v \| \leqslant \| df^t v \| \leqslant c\mu_2^t \| v \|, \quad v \in E_x^0,$$
$$c^{-1}\lambda_3^t \| v \| \leqslant \| df^t v \| \leqslant c\mu_3^t \| v \|, \quad v \in E_x^+.$$

The distribution E^- (E^+) is integrable.

10. Topological and ergodic properties.
When we speak of topological properties, we have in mind such properties as *topological transitivity, topological mixing, minimality*. By ergodic properties we mean *ergodicity, mixing, the K-property, the Bernoulli property* (isomorphism to a Bernoulli flow). We need also the concepts of *topological and metric entropy*. Definitions of the properties and concepts can be found, for example, in the survey [20], and a more detailed account is in the book mentioned at the beginning of this section.

§2. Local theory

1. A *geodesic flow* M on a manifold is a *flow* in TM whose action is given by the formula

$$f^t(x, v) = \dot{\gamma}_v(t), \quad x \in M, \quad v \in T_xM.$$

It belongs to the class C^{r-1}, where r is the smoothness class of the Riemannian metric. We denote by V the vector field specifying f^t. We describe another way of defining V suggested by classical mechanics.

We consider the *cotangent bundle* T^*M whose elements are the linear 1-forms on the spaces tangential to M and let π^*: $T^*M \to M$ be the natural projection. We define on T^*M the *canonical* 1-form ω by

$$\omega(x, q) = q(d\pi^*(x,q)), \; x \in M, \; q \in T_x^*M$$

(note that $d\pi^*(x, q) \in T_xM$), and set $\Omega_* = d\omega$. The *canonical 2-form* Ω_* is non-degenerate and gives on T^*M a *symplectic structure* (see [6], Ch. 8 and [21], Ch. 3). The Riemannian metric allows us to define the identity map[(1)] $\mathcal{L}$: $TM \to T^*M$

$$\mathcal{L}(x, v) = (x, q), \ x \in M, \ v \in T_xM, \ q \in T_x^*M,$$

where the 1-form q satisfies the relation $\langle v, w\rangle = q(w)$ for any $w \in T_xM$. We now give the *canonical 2-form* Ω in TM by setting

$$\Omega(Y, Z) = \Omega_*(\mathcal{L}(Y), \mathcal{L}(Z)), \ Y, Z \in TM.$$

We define a function K: $TM \to M$ by

$$K(x, v) = \frac{1}{2}\langle v, v\rangle = \frac{1}{2}\| v \|^2.$$

The vector field V giving the geodesic flow can now be defined as the field corresponding to the 1-form dK with respect to the canonical 2-form Ω, so that for any $x \in M$, $v \in T_xM$, and $z \in T_{(x,v)}TM$

$$\Omega(V(x, v), Z) = dK(Z) \tag{2.1}$$

(see [6], [25]).

2. **The connection with classical mechanics** (see [6], [4], [25], [21]). The approach described above is connected with an interpretation of geodesic flows as mathematical models of classical mechanics, that is, as the phase flows of certain Hamiltonian systems. Here the manifold M plays the role of a configuration, and TM is the phase space of the mechanical system; points x from M are treated as "generalized" coordinates, vectors v from TM as "generalized" velocities, and elements from T^*M as "generalized" impulses. In accordance with Hamilton's principle, a trajectory of the mechanical system in the configuration space passing through x, y is the extremal of the energy functional (in more classical terminology: the action $E = \frac{1}{2}\int_{t_0}^{t_1} \langle v(t), v(t)\rangle dt$ (there is no potential energy). Here, $\|v(t^0)\| = const$ along the extremal. It is well known that these extremals are geodesics, that is, the equation of the geodesics is the Euler equation for the corresponding variational problem with fixed end-points.

For a description of the trajectories of a mechanical system in the phase space we look again at $K(x, v)$, which is also called the Hamiltonian. A

(1) The map $\mathcal{L}$ is a particular case of the Legendre transformation, which is defined not only for Riemannian metrics, but also in a more general situation including Finsler metrics, which makes it possible to study geodesic flows in the relevant manifolds.

geodesic flow is a phase flow of the corresponding Hamiltonian system[1] (that is, a Hamiltonian flow), and its vector field in local co-ordinates (x, v) has the form $\left(\frac{\partial K}{\partial v}, -\frac{\partial K}{\partial x}\right)$.

Since the total energy of the system is a first integral, the surfaces $K(x, v) = const$, (that is, $\|v\| = const$), are invariants of the geodesic flow; therefore, we usually consider the geodesic flow on the manifold (which we denote by SM) of the unit linear elements (that is, the pairs (x, v), $x \in M$, $v \in T_xM$, $\|v\| = 1$), which is a smooth fibration with basis M and the $(p-1)$-dimensional unit sphere as fibre.

This representation allows us to define an invariant measure μ for a geodesic flow: in accordance with Louiville's theorem, this measure is given by

$$d\mu = d\sigma\, d\nu,$$

where $d\sigma$ is a surface element on the $(p-1)$-dimensional unit sphere.

3. Variational equations.

Let $v \in TM$ and $\xi \in T_vSM$. We associate with a vector ξ the Jacobi field Y_ξ given by the initial conditions

$$Y_\xi(0) = d\pi\xi,\ Y'_\xi(0) = K\xi. \tag{2.2}$$

Proposition 2.1 (see [53], Proposition 1.7). *The map $\xi \to Y_\xi$ is a linear isomorphism of T_vSM onto $J(\gamma_v)$.*

2) $Y_\xi(t) = d\pi \circ df^t\xi$, $Y'_\xi(t) = K \circ df^t\xi$ *for every* $t \in \mathbf{R}$.

3) *For $v \in SM$ the map $\xi \to Y_\xi$ is a linear isomorphism of $V(v)^\perp$ (the orthogonal complement to $V(v)$ in T_vSM) onto $J_0(\gamma_v)$, that is, $\xi \in T_vSM$, and $\langle \xi, V(v)\rangle = 0$ if and only if the Jacobi field $Y_\xi(t)$ is orthogonal.*

Corollary 2.1. 1) *If $v \in SM$, $\xi \in T_vSM$, and $\langle \xi, V(v)\rangle = 0$, then $\langle df^t\xi, V(f^t(v))\rangle = 0$ for all $t \in \mathbf{R}$.*

2) *If $v \in TM$ and $\xi \in T_vTM$, then $\| df^t\xi \|^2 = \| Y_\xi(t) \|^2 + \|Y'_\xi(t)\|^2$ (the norm induced by the canonical Riemannian metric in TM, see* §1.1).

[1] Hamiltonian systems of a more general form are defined on an arbitrary symplectic manifold N (that is, a manifold on which the symplectic 2-form Ω is given with the help of a certain Hamiltonian function H: $N \to \mathbf{R}$. The vector field specifying the Hamiltonian flow is a field corresponding to the 1-form dH relative to the 2-form Ω (see (2.1)). In the local coordinates (x, p) in which $\Omega = \sum dp_i \wedge dx^i$ this vector field has the form $(\partial H/\partial p, -\partial H/\partial x)$. For a conservative mechanical system with a configuration manifold M we have $N = T^*M$ (with corresponding Ω), $H = K - U$, where K is the kinetic and U is the potential energy of the system. For a fixed value E of the Hamiltonian function H, the motion can be reduced to a geodesic flow on M by the introduction of a new Riemannian metric $\langle\ ,\ \rangle'_x = (E - U(x))\langle\ ,\ \rangle'_x$ and a certain change of time (the Maupertuis-Lagrange-Jacobi principle). In our case all this simplifies since $U = 0$ and T^*M is naturally isomorphic to TM.

From these propositions it follows that, (to within a correspondence $Y_\xi \to (Y_\xi, Y'_\xi)$) the Jacobi equation (1.1) (or in the form (1.2)) is the variational equation for the geodesic flow (compare with (1.7)). This assertion has undoubtedly been known for a long time. In [3] a proof is given by means of direct calculations using local coordinates; another method of argument in "invariant" terms is given in [21], §3.

4. Limiting solutions.
To begin with, we give one proposition, which plays an important role in the study of the properties of solutions of Jacobi and Riccati equations on manifolds without conjugate points.

Proposition 2.2 (see [53], Lemma 2.8). *Let M be a complete Riemannian manifold without conjugate points for which the curvatures $K_x(P)$ are uniformly bounded below by $-k^2$, $k > 0$. Then $v \in \mathbf{R}^{p-1}$, $\|v\| = 1$ for any symmetric solution $U(t)$ of the Riccati equation* (1.4), *and for any $t > 0$*

$$\langle U(t)v, v\rangle \leqslant k \coth(kt).$$

For manifolds of negative curvature the principal fact of the boundedness of $U(t)$ (but without this estimate) was proved in [3] and in other terms in [4].

Corollary 2.2 (see [53], Proposition 2.7). *Under the conditions of Proposition* 2.2, *for any orthogonal Jacobi field $Y(t)$ along a geodesic $\gamma(t)$ such that $Y(0) = 0$,*

$$\| Y'(t) \| \leqslant k \coth(kt) \| Y(t) \|.$$

We consider the Jacobi matrix equation (1.3) and let $D_s(t)$, $s > 0$, be its solution subject to the boundary conditions $D_s(0) = I$, $D_s(s) = 0$.

Theorem 2.1. *Under the conditions of Proposition* 2.2 *there exists a solution $D^-(t)$ of* (1.3) *with the initial conditions*

$$D^-(0) = I, \quad \frac{d}{dt} D^-(t)\Big|_{t=0} = \lim_{s\to\infty} \frac{d}{dt} D_s(t)\Big|_{t=0}.$$

This solution has the following properties:
1) *it is non-degenerate, that is, for every $t \in \mathbf{R}$*

(2.3) $$\det(D^-(t)) \neq 0;$$

2) $D^-(t) = \lim_{s\to\infty} D_s(t)$ *for every* $t \in \mathbf{R}$;
3) *for every $t > 0$*

$$D(t) = A(t) \int_t^\infty A^{-1}(s) (A^{-1}(s))^* \, ds,$$

where $A(t)$ is the solution of (1.3) *with the initial conditions $A(0) = 0$, $\frac{d}{dt} A(t) \mid_{t=0} = I$.*

This $D^-(t)$ is called *the negative limiting solution* of (1.3). A *positive limiting solution* $D^+(t)$ is constructed similarly (in the preceding constructions, $s \to -\infty$). The existence of limiting solutions was established by Green in [66] and [67] and reproduced by Eberlein in [53]; for two-dimensional manifolds without focal points a corresponding (but naturally simpler) argument was given in [23] (see also [64], §1).

5. Invariant distributions.
For $v \in SM$ we put

$$X^-(v) = \{\xi \in T_vSM : \langle \xi,\ V(v)\rangle = 0,\ Y_\xi(t) = D^-(t)d\pi\xi\},$$
$$X^+(v) = \{\xi \in T_vSM: \langle \xi,\ V(v)\rangle = 0,\ Y_\xi(t) = D^+(t)d\pi\xi\}.$$

Then $X^-(v)$ and $X^+(v)$ are called stable and unstable subspaces of T_vSM, respectively. We write

$$v^\perp = \{w \in T_{\pi(v)}M: w \text{ orthogonal to } v\}$$

and let τ: $SM \to SM$ be the involution $\tau(v) = -v$.

Theorem 2.2 (see [53], §2). *Suppose that the manifold M is complete and the curvatures $K_x(P)$ are uniformly bounded below by $-k^2$, $k > 0$. Then*

1) *$X^-(v)$ and $X^+(v)$ are vector subspaces of T_vSM of dimension $p-1$;*
2) $df^tX^-(v) = X^-(f^t(v))$, $df^tX^+(v) = X^+(f^t(v))$;
3) $d\pi X^-(v) = d\pi X^+(v) = v^\perp$;
4) *if* $\xi \in X^-(v)$ *or* $\xi \in X^+(v)$, *then* $Y_\xi(t) \neq 0$;
5) $X^+(-v) = d\tau X^-(v)$, $X^-(-v) = d\tau X^+(v)$;
6) $\| K\xi \| \leqslant k \| d\pi\xi \|$ *for any* $v \in SM$, $\xi \in X^-(v)$ *or* $\xi \in X^+(v)$;
7) *if* $v \in SM$, $\xi \in T_vSM$, $\langle \xi,\ V(v)\rangle = 0$, *and* $\| d\pi \circ df^t\xi \| \leqslant \text{const}$ *for all* $t \geqslant 0$ (*or* $t \leqslant 0$), *then* $\xi \in X^-(v)$ (*or* $\xi \in X^+(v)$).

Here 1) means that the collection of subspaces $X^-(v)$ and $X^+(v)$ form two $(p-1)$-dimensional distributions in TSM; 2) means that these distributions are invariant under the geodesic flow. From 3) it follows that for any $w \in v^\perp$ there is a unique "*stable limiting*" Jacobi field $Y_w^-(t)$ for which $Y_w^-(0) = w$ and $(Y_w^-)'(0) = K\xi$, where ξ is the uniquely determined vector from $X^-(v)$, for which $d\pi\xi = w$. An "*unstable limiting*" Jacobi field $Y_w^+(t)$ can be constructed similarly. For manifolds of negative curvature (and in some other cases) these fields can be defined by the boundary conditions $Y_w^-(0) = w$ and $Y_w^-(+\infty) = 0$ (or $Y_w^+(0) = w$ and $Y_w^+(-\infty) = 0$, respectively). By virtue of 4) (see also (2.3)) the Jacobi fields $Y_w^+(t)$ and $Y_w^-(t)$ are non-singular.

For stable limiting Jacobi fields on manifolds of non-positive curvature the following comparability theorem holds.

***Theorem* 2.3** (see [75], Theorem 2.4). *Let M be a complete Riemannian manifold such that $-b^2 \leqslant K_x(P) \leqslant -a^2$ for all curvatures $K_x(P)$. Then for any $v \in SM$ and $w \in v^\perp$, the stable limiting Jacobi field $Y_w^-(t)$ along the geodesic $\gamma_v(t)$ satisfies the inequalities*

$$\| w \| e^{-bt} \leqslant \| Y_w^-(t) \| \leqslant \| w \| e^{-at}.$$

We lift the distributions X^- and X^+ to the universal Riemannian covering manifold H (and denote the resulting distributions by the same letters; see §1.3). Then

$$d(d\varphi)X^-(v) = X^-(d\varphi v),\ d(d\varphi)X^+(v) = X^+(d\varphi v), \tag{2.4}$$

where φ is an isometry of H.

The existence of the subspaces $X^-(v)$ and $X^+(v)$ confirms the view that a geodesic flow on a manifold without conjugate points inherits certain hyperbolic properties. From Theorem 2.2.7) it follows that a vector $\xi \in T_vSM$ for which $\chi^+(v, \xi) < 0$ (or $\chi^+(v, \xi) > 0$) lies in $X^-(v)$ (or $X^+(v)$). Of course, a vector $\xi \in T_vSM$ for which(1) $\chi^+(v, \xi) = 0$ may lie in both $X^-(v)$ and $X^+(v)$. In particular the intersection of $X^-(v)$ and $X^+(v)$ need not be trivial.(2)

It is not known whether the distributions X^- and X^+ are continuous. The most general sufficient conditions known at present are given by Theorem 5.6.

On manifolds without focal points X^- and X^+ have certain supplementary properties.

***Proposition* 2.3** (see [53], §2 and [34], §4). *Suppose that under the conditions of Theorem 2.2 a Riemannian metric in M has no focal points.*

1) *If $\xi \in X^-(v)$ (or $\xi \in X^+(v)$), then the function $\varphi(t) = \| Y_\xi(t) \|$ is non-increasing (non-decreasing);*

2) *if $v \in SM$, $\xi \in T_vSM$, and $\langle \xi, V(v) \rangle = 0$, then $\| d\pi \circ df^t\xi \| \leqslant \mathrm{const}$ for all $t \geqslant 0$ (or $t \leqslant 0$) if and only if $\xi \in X^-(v)$ (or $\xi \in X^+(v)$).*

§3. Hyperbolic properties of geodesic flows

1. Manifolds of negative curvature.

Geodesic flows on manifolds of negative curvature have extremely strong hyperbolic properties. This follows from a remarkable theorem that was proved by Anosov in [3] (another proof is in the article by Asonov and Sinai [4]).

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(1)For such vectors the quantity $\| d\pi \circ df^t\xi \|$ can be unbounded (but the rate of growth must be less than exponential); therefore, the property established by 7) is sufficient but, generally speaking, not necessary (it is so if the metric has no focal points, see Proposition 2.3).

(2)Compare with Theorem 8.4.

***Theorem* 3.1.** *A geodesic flow on a manifold whose curvature is negative and bounded below is a Y-flow. Moreover,*

$$E_v^- = X^-(v), \; E_v^+ = X^+(v)$$

(in particular, $X^-(v) \cap X^+(v) = 0$ *and* (1.8) *holds for any* $\xi \in X^-(v)$ *or* $\xi \in X^+(v)$.

The second assertion follows directly from Theorem 2.3 (in essence, the necessary reasonings are in [4]; see also [21] and [53]).

The proof of Theorem 3.1 is based on the argument that if along a given geodesic γ the curvature at each point $\gamma(t)$ and in each two-dimensional direction is negative (and bounded below), then along the corresponding trajectory of the geodesic flow the conditions of uniform hyperbolicity are fulfilled. Thus, these arguments are, so to speak, trajectorial: they operate only with one unique trajectory, and uniformity is achieved at the expense of uniform bounds on the curvature. This approach has far-reaching generalizations and allows us to obtain non-uniform estimates when the geodesic from time to time goes through parts with positive curvature.

2. Manifolds without focal or conjugate points.
Let $v \in SM$. We choose an arbitrary *system of Fermi coordinates along a geodesic* $\gamma_v(t)$ and a vector w orthogonal to v. We set

$$w(t) = \frac{D^-(t)\,w}{\|D^-(t)\,w\|}, \qquad K_{v,w}(t) = \langle R(\dot\gamma_v(t), \; w(t))\,\dot\gamma_v(t), \; w(t)\rangle,$$

where $D^{-1}(t)$ is the orthogonal limiting solution of the Jacobi equations (1.3). We consider the set

$$\Lambda_0 = \Big\{ v \in SM\colon \; w \text{ is orthogonal to } v, \text{for any } w \in SM, \text{ and } \overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t K_{v,w}(s)\,ds < 0 \Big\}.$$

The following theorem, which is proved in [32] (Theorem 10.5) describes the hyperbolic properties of a geodesic flow on Λ_0.

***Theorem* 3.2.** *Suppose that a Riemannian manifold M has no focal points. Then* $\chi^+(v, \xi) < 0$, *(or* $\chi^+(v, \xi) > 0$) *for any* $v \in \Lambda_0$ *and* $\xi \in X^-(v)$ *(or* $\xi \in X^+(v)$).

In the two-dimensional case this theorem has a converse.

***Theorem* 3.3** (see [34], Theorem 8.2). *Suppose that a Riemannian manifold M does not have conjugate points and that* $\dim M = 2$. *Let* $v \in SM$ *be such that* $\chi^+(v, \xi) < 0$ *for some (and consequently for any)* $\xi \in X^-(v)$. *Then* $v \in \Lambda_0$.

One can speak of hyperbolic properties of a geodesic flow only if the set Λ defined by (1.10) has positive measure. (It can be shown under certain supplementary assumptions that in this case $\mu(\Lambda) = 1$.) We quote sufficient conditions for $\mu(\Lambda) > 0$.

***Theorem* 3.4** (see [34], §9 and [32], §10). 1) *If a Riemannian metric does not have focal points and* $\mu(\Lambda_0) > 0$, *then* $\mu(\Lambda) > 0$.

2) *If* $\dim M = 2$, *the Euler characteristic* $\chi(M)$ *is negative, and the Riemannian metric does not have focal points, then* $\mu(\Lambda_0) > 0$.

3) *If* $\dim M = 2$, *the Riemannian metric does not have conjugate points, and the entropy of the geodesic flow* $h_\mu(f)$ *is positive, then* $\mu(\Lambda) > 0$.

§4. The axiom of visibility and the axiom of asymptoticity

1. Definition of the axiom of visibility.

At the beginning of the 70's Eberlein introduced and systematically studied in a group of papers [50]–[52] (see also [106]) a condition on a Riemannian metric which he called the *axiom of visibility*. Originally, this condition arose from attempts to find a geometric analogue for the "cosine law" and for a number of other properties inherent in a metric with non-negative curvature (see below §4.6). However, a generalization which Eberlein found turned out to be exceptionally successful and productive. It became clear that this condition concerns not only the given Riemannian metric but also the whole class of Riemannian metrics without conjugate points (see Theorem 4.2); to some extent it reflects the topological properties of the manifold as such.(1) Moreover, it ensures the topological transitivity of a geodesic flow (see Theorem 6.1) and, as was established later, it has a relation to its hyperbolic properties. As we have already shown, the class of Riemannian metrics without conjugate points is a natural generalization, from the point of view of the preservation of hyperbolic properties, of the class of Riemannian metrics with negative curvature. Nevertheless, it is fairly broad in so far as it contains, for example, Riemannian metrics of zero curvature,(2) in which a geodesic flow does not have any hyperbolic properties (for such metrics $X^-(v) = X^+(v)$, $v \in SM$). Therefore, some additional conditions (excluding specifically the case of Riemannian metrics of zero curvature) are required, one of which is precisely Eberlein's condition.

The axiom of visibility is stated for a given Riemannian metric in terms of the universal covering Riemannian manifold H (in other words, for simply-

(1) Although the connection between the axiom of visibility and topology has not been clarified directly, one can obtain some information in, for example, the two-dimensional case: the presence of a Riemannian metric without conjugate points singles out from among all surfaces those with the Euler characteristic $\chi(M) \leqslant 0$, and the axiom of visibility those with $\chi(M) < 0$.

(2) We mention in this context, one result of Hopf [78] to the effect that on a two-dimensional torus any metric without conjugate points has zero curvature.

connected Riemannian manifolds) and it consists in the following:

for any $x \in H$ *and* $\varepsilon > 0$ *there is an* $R = R(x, \varepsilon)$ *such that for every geodesic segment* $\gamma(t)$, $t_0 \leqslant t \leqslant t_1$, *in* H *for which* $\rho(x, \gamma) \geqslant R$ *we have*

$$\measuredangle_x(\gamma(t_0), \gamma(t_1)) \leqslant \varepsilon,$$

($\measuredangle_x(a, b)$ denotes the angle between the vectors $\dot{\gamma}_{xa}(0)$ and $\dot{\gamma}_{xb}(0)$). A manifold H is said to satisfy the *axiom of uniform visibility* if[(1)] the constant R can be chosen independent of x.

We say that a connected (not necessarily simply-connected) manifold M satisfies *the axiom of* (*uniform*) *visibility* if its universal covering Riemannian manifold satisfies this axiom.

The following theorem gives sufficient conditions under which a Riemannian manifold satisfies the axiom of uniform visibility.

Theorem 4.1 (see [106], Proposition 5.9 and [50], Theorem 4.1).

1) *A complete Riemannian manifold for which* $K_x(P) \leqslant c \leqslant 0$ *for all* x *and* P *satisfies the axiom of uniform visibility.*

2) *A simply-connected Riemannian manifold with a compact factor for which* $K_x(P) \leqslant 0$ *for all* x *and* P *satisfies the axiom of uniform visibility if and only if it does not admit of a global geodesically isometric embedding of the Euclidean plane* $\mathbf{R}^2$ (*in particular, plane manifolds do not satisfy the axiom of visibility*).[(2)]

3) *For simply-connected manifolds with a compact factor for which* $K_x(P) \leqslant 0$ *for all* x *and* P, *the axioms of visibility and of uniform visibility are equivalent.*

The following important proposition shows that the axiom of uniform visibility characterizes at one stroke the whole class of Riemann metrics without conjugate points.

Theorem 4.2 (see [50], Theorem 5.1). *Let* M *be a compact manifold with a Riemannian metric without conjugate points satisfying the axiom of uniform visibility. Then any metric in* M *without conjugate points also satisfies this axiom.*

To prove this theorem we use one proposition of independent interest.

Theorem 4.3. *Let* $\langle , \rangle$ *and* $\langle , \rangle^*$ *be two equivalent Riemannian metrics*[(3)] *on a simply-connected manifold* H *satisfying the axiom of uniform visibility* (*not necessarily without conjugate points*). *Then there is an* $R > 0$ *such that for*

[(1)]These definitions do not formally assume the absence of conjugate points, indeed some of the results quoted below can be obtained without this assumption.

[(2)]That is, an embedding of $\mathbf{R}^2$ under which the complete geodesic on the corresponding two-dimensional submanifold is at the same time a geodesic on the manifold.

[(3)]That is, $a\|v\| \leqslant \|v\|^* \leqslant b\|v\|$ for some $a > 0$, $b > 0$ and any $v \in TM$.

any two geodesic segments γ and γ^* (in the metrics $\langle , \rangle$ and $\langle , \rangle^*$, respectively), with the same end-points and with γ^* minimizing[1] *for any $x \in \gamma$*

$$\rho(x, \gamma^*) \leqslant R.$$

This assertion was proved by Morse in [93] for the case when $\langle , \rangle^*$ is the standard metric on the Lobachevskii plane, and by Klingenberg in [83] for arbitrary manifolds of negative curvature (in [21] the case of a manifold of hyperbolic type is examined, see also [61]). In the general case a proof was given by Eberlein in [50] (see Proposition 5.5).

***Theorem* 4.4.** *Let M be a two-dimensional compact Riemannian manifold with Euler characteristic $\chi(M) < 0$ and with a Riemannian metric without conjugate points. Then M satisfies the axiom of uniform visibility.*

This follows directly from Theorem 4.2, since any compact two-dimensional manifold with $\chi(M) < 0$ admits of a Riemannian metric of negative curvature.

2. The absolute.

Two geodesics γ_1 and γ_2 on a simply-connected manifold H are said to be *positively asymptotic* (or simply *asymptotic*)[2] if

$$\rho(\gamma_1(t), \gamma_2(t)) \leqslant \text{const} \tag{4.1}$$

for all $t > 0$, and *negatively asymptotic* if (4.1) holds for all $t < 0$. The relation of asymptoticity is an equivalence relation. The class of geodesics positively (or negatively) asymptotic to a geodesic $\gamma(t)$ is denoted by $\gamma(+\infty)$ (or $\gamma(-\infty)$) and is called a *point at infinity*. Since $\gamma_v(-\infty) = \gamma_{-v}(+\infty)$, the sets of equivalence classes of positively and negatively asymptotic geodesics are equal and form an *absolute* of H, denoted by $H(\infty)$. We put $\bar{H} = H \cup H(\infty)$. Although these definitions are applicable to any Riemannian metric one can gain substantial information on the structure and properties of an absolute only under an additional assumption on the metric. We quote here one of the most general conditions of this kind, which was introduced in [34].

3. Definition of the axiom of asymptoticity.

Apart from the axiom of visibility there are also other stronger or weaker conditions on a Riemannian metric with an analogous role: they allow us to study geometric properties of a manifold (with a given Riemannian metric). We formulate one of these conditions whose significance will be made clear below. We recall that we only consider Riemannian metrics without conjugate points.

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(1) That is, $\rho^*(x, y) = |t_1 - t_2|$ for any $x, y \in \gamma^*$, $x = \gamma^*(t_1)$, $y = \gamma^*(t_2)$.

(2) Thus, contrary to etymology, parallel straight lines in the Euclidean plane are asymptotic.

We choose an arbitrary point x on a simply-connected Riemannian manifold H and a vector $v \in SH$. Let $x_n \in H$, $x_n \to x$, $v_n \in SH$, $v_n \to v$ and $t_n \to +\infty$. Let γ_n be the geodesic joining the points x_n and $\gamma_{v_n}(t_n)$. The sequence of vectors $\dot\gamma_n(0)$ is compact, consequently, the sequence of geodesics has a limiting geodesic.(1) We say (see [34]) that the Riemannian metric in H (and, loosely, H itself) satisfies the *axiom of asymptoticity* if for any choice of x_n, $x \in H$, $x_n \to x$, v_n, $v \in SH$, $v_n \to v$ and for $t_n \to +\infty$, any limiting geodesic of the sequence γ_n is positively asymptotic to the geodesic γ. From Proposition 1.3, 3) it follows that in this case the sequence of geodesics γ_n has a single limit geodesic, that is, it converges.

We say that a connected (not necessarily simply-connected) manifold satisfies the *axiom of asymptoticity* if its universal Riemannian covering manifold does.

Theorem **4.5** (see [34], Theorem 5.1 and Proposition 5.4). *A complete simply-connected Riemannian manifold H satisfies the axiom of asymptoticity if one of the following conditions holds*: 1) *the Riemannian metric does not have focal points*;(2) 2) *H satisfies the axiom of uniform visibility*;(3) 3) $\dim M = 2$, $\chi(M) < 0$.

4. The topology of an absolute.

Let $x \in H$, $v \in SH$, $\pi(v) = x$. We call the set

$$C(v,\ \varepsilon) = \{y \in \bar H : \measuredangle_x(\gamma_v(+\infty),\ y) < \varepsilon\}$$

a *cone* in H with vertex at x, axis v and angle ε, and the set $T(v,\ \varepsilon\ r) = C(v,\ \varepsilon) \setminus \{y \in H : \rho(x,\ y) \leqslant r\}$ a *truncated cone* in $\bar H$ with vertex at x, axis v, angle ε and radius r.

Theorem **4.6.** *Let H be a simply-connected Riemannian manifold with a compact factor and a Riemannian metric satisfying the axiom of asymptoticity. Then a topology τ can be introduced on $\bar H$ such that*

1) *the restriction of τ to the space H is the same as the topology on H induced by the Riemannian metric*;

2) *the set H is open and dense in $\bar H$; the sets $H(\infty)$ and $\bar H$ are compact*;

3) *for each $p \in H(\infty)$ the set of cones containing p is a local basis for the topology τ at p*;

4) *for any $p \in H(\infty)$, $x \in H$, the set of truncated cones with vertex x containing p is a local basis for the topology τ at p*;

(1)A sequence of geodesics γ_n tends to γ if $\gamma_n(0) \to \gamma(0)$ and $\dot\gamma_n(0) \to \dot\gamma(0)$.

(2)This is proved in [34] under the additional assumption that all curvatures $K_x(P)$ are uniformly bounded below. Using results of Goto (see [64], Theorem 1) and O'Sullivan (see [103], Proposition 1) it can be proved without this assumption.

(3)We mention a consequence of the axiom of visibility, which establishes a property close to the axiom of asymptoticity (but stronger; see [50], Lemma 1.6): if x_n, y_n, $z_n \in H$, $x_n \to x$, $y_n \to y$ and z_n diverges, if γ_n and σ_n are geodesics joining the points x_n, z_n and y_n, z_n, respectively, and if γ and σ are the limiting geodesics of the sequences γ_n and σ_n, then γ and σ are asymptotic.

5) *for any* $x \in H$, *the map* φ_x: $\overline{B} \to \overline{H}$ (*B is the open unit ball in* T_xM) *defined by*

$$\varphi_x(y) = \begin{cases} \exp_x[(1-\|v_y\|)^{-1}], & y \in B, \\ \gamma_{v_y}(+\infty), & y \in \partial B \end{cases}$$

(*where* v_y *is the vector in* T_xM *with initial point at zero and end-point at* y) *is a homeomorphism.*

For the Lobachevskii space this was already known to Poincaré, and for the case of variable negative curvature it is in Busemann's book [13]. For complete (not necessarily compact) manifolds without focal points, in dimension 2 or on the supposition that the manifold is of Anosov type, this result was obtained by Goto in [64] (see Theorems 5 and 6) and was announced by her for any complete manifold without focal points). For complete simply-connected manifolds satisfying the axiom of uniform visibility it was established by Eberlein in [50] (see Proposition 1.12 and 1.13). In full generality it was proved in [34] (see Theorem 7.6). It can also be proved that the topology τ is *admissible* in the sense of [106], p.50.

The following proposition is an important consequence of the axiom of asymptoticity.

Proposition 4.1 (see [34], Proposition 5.1). *If a simply-connected Riemannian manifold satisfies the axiom of asymptoticity, then for any geodesic* γ *and for any point* $x \in H$ *there is a geodesic* γ' *through* x *and asymptotic to* γ. *This geodesic is uniquely determined if H has a compact factor or, more-generally, if all the curvatures* $K_x(P)$ *are uniformly bounded below* (*see Proposition* 1.3, 3) *and Remark* 1.1).

Special cases of the proposition are in Eberlein [50] for manifolds of non-positive curvature or satisfying the axiom of visibility, and in Goto [64] and O'Sullivan [103] for manifolds without focal points (and even without the assumption that the curvatures are bounded, see footnote(2) on p.23).

Proposition 4.1 allows us to define the following functions:

1) $\Psi_t : SH \to \overline{H}$, $\Psi_t(v) = \gamma_v(t)$, $t \in [0, \infty]$;

2) $\varphi : H \times H(\infty) \to SH$, $\varphi(x, p) = \dot{\gamma}_{xp}(0)$.

Proposition 4.2 (see [34], Corollaries 7.3 and 7.4). *The functions* Ψ_t *and* φ *are continuous.*

5. Elements of uniqueness. The strong axiom of uniform visibility.

In a complete simply-connected Riemannian manifold any two points can be joined by a geodesic (see [29], the Rhinov-Hopf theorem). This is uniquely determined if the Riemannian metric does not have conjugate points (see Proposition 1, 2)). As Eberlein has shown in [50], the axiom of visibility makes it possible to extend these propositions to the space $\overline{H}$.

***Theorem* 4.7** (see [50]). *Let H be a complete simply-connected manifold with a Riemannian metric without conjugate points and satisfying the axiom of uniform visibility. Then for any $p, q \in H(\infty)$ there is a geodesic γ such that*

$$\gamma(+\infty) = p, \ \gamma(-\infty) = q. \tag{4.2}$$

However, such a geodesic need not be unique. In this context we introduce the following concepts. We call a vector $v \in SH$ an *element of non-uniqueness* if there exists a vector $w \in SH$ such that

$$\gamma_v(+\infty) = \gamma_w(+\infty), \ \gamma_v(-\infty) = \gamma_w(-\infty). \tag{4.3}$$

Otherwise, $v \in SH$ is called an *element of uniqueness*. We say that a Riemannian metric without conjugate points satisfies the *strong axiom of uniform visibility* if it satisfies the axiom of uniform visibility and for any $p, q \in H(\infty)$ there exists a unique geodesic γ satisfying (4.2); an equivalent definition is that every $v \in SH$ is an element of uniqueness.

In [106] (see Corollary 5.2) it is proved that a manifold of (strictly) negative curvature satisfies the strong axiom of visibility.

***Theorem* 4.8.** *Let H be a simply-connected Riemannian manifold without focal points. The following assertions are equivalent.*

1) *H does not satisfy the strong axiom of visibility.*

2) *There exists a globally geodesically isometric embedding*[1] *of the strip $\{(x, y) : 0 \leqslant x \leqslant c, -\infty < y < \infty\}$ in H for some $c > 0$.*

3) *There exists a geodesically isometric embedding of the rectangle $\{(x, y) : 0 \leqslant x \leqslant c, 0 \leqslant y \leqslant T\}$ in H for some $c > 0$ and any $T > 0$.*

This proposition is called "the theorem of the plane strip". For manifolds of non-positive curvature it was proved by Eberlein in [50] (see Proposition 5.1); for manifolds without focal points in the two-dimensional case by Green in [67], and under the assumption that H has a compact factor, in [34]; finally, for arbitrary manifolds without focal points by Eschenburg in [60] (the same result was also found by O'Sullivan in [103] under some additional assumptions which are easily removed with the help of Goto's results [64]). Theorem 4.8, 2) can be sharpened somewhat.

***Proposition* 4.3** (see [34], Theorem 7.3). *Suppose that a manifold H satisfies the conditions of Theorem 4.8 and has a compact factor, and that $v \in SH$ is an element of non-uniqueness. Then there exists a vector $w \in SH$ orthogonal to v and a geodesic segment $\delta(s)$, $0 \leqslant s \leqslant a$, such that $\delta(0) = \pi(v)$, $\dot\delta(0) = w$, and the union of the geodesics passing orthogonally through the points of $\delta(s)$ is the image under the globally geodesically isometric embedding of the strip $\{(s, t) \in \mathbf{R}^2 : 0 \leqslant s \leqslant a, -\infty < t < \infty\}$.*

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[1] See footnote to Theorem 4.1.

This assertion has an interesting consequence about the totalities of all geodesics joining two given points p and q on the absolute $H(\infty)$, which was established in [106] for manifolds of non-positive curvature.

***Proposition* 4.4.** *Suppose that a manifold H has a compact factor and satisfies the conditions of Theorem 4.8, and that $p, q \in H(\infty)$. Then there exists a compact set $K = K(p, q)$ such that any geodesic joining p and q intersects K.*

6. Manifolds of non-positive curvature. Classification of isometries. An Hadamard manifold is a complete simply-connected Riemannian manifold of non-positive curvature. For such manifolds the "cosine law" holds: $a^2 + b^2 - c^2 \leqslant 2ab \cos\theta$, where a, b, and c are the sides of a geodesic triangle and θ is the angle formed by a and b. Using this property only, one can advance a long way in the investigation of the geometry of the manifold H, in particular, construct an absolute and prove the propositions stated in Theorem 4.6 (see [106], §§1 and 2). Besides this, it follows from this fact that Hadamard manifolds satisfy the axiom of asymptoticity (see Theorem 4.5). Deeper information can be obtained if we suppose that these manifolds satisfy the axiom of uniform visibility, which in the given case is equivalent to the assertion that any two points on the absolute $H(\infty)$ can be joined by a geodesic (see [106], §4; compare Theorem 4.7).

The axiom of visibility allows us to classify the isometries on an Hadamard manifold: every isometry φ gives rise to a map of $\bar{H}$, and the classification is based on the number of fixed points of φ. This approach was suggested by Poincaré's work on the theory of automorphic functions and, in particular, on his study of the linear fractional transformations of the Lobachevskii plane (the Poincaré model). We give a short summary of the relevant results from [50], [51], and [106].

Let φ be an isometry of an Hadamard manifold H. The action of φ can be extended to the absolute $H(\infty)$, by putting

$$\varphi(x) = (\varphi\circ\gamma)\ (+\infty), \tag{4.4}$$

where $x \in H(\infty)$ and γ is any geodesic for which $\gamma(+\infty) = x$ (the result does not depend on the choice of γ). Since $\bar{H}$ is compact, the map φ has a fixed point.

***Theorem* 4.9.** *For each isometry φ on an Hadamard manifold H satisfying the axiom of visibility only one of the following three possibilities can occur:*

1) *φ has at least one fixed point belonging to H;*

2) *φ has exactly one fixed point belonging to $H(\infty)$ and has no fixed points in H;*

3) *φ has exactly two fixed points belonging to $H(\infty)$ and has no fixed points in H.*

In the first case the isometry is said to be *elliptical*, in the second *parabolic*, and in the third *axial.* Some properties of axial isometries are described in the following theorem.

***Theorem* 4.10.** 1) *An isometry* φ *is axial if and only if there exists a geodesic* γ *such that* $(\varphi\circ\gamma)(t) = \gamma(t+a)$, $a > 0$, *(this geodesic is called the axis of the isometry); here* $\gamma(-\infty) = x$, $\gamma(+\infty) = y$, *where* x *and* y *are the fixed points of* φ.

2) *If* γ *is the axis of an axial isometry* φ, *then for any* $x \in H$, *under a suitable orientation,* $\varphi^{-n}(x) \to \gamma(-\infty)$, $\varphi^{n}(x) \to \gamma(+\infty)$, *as* $n \to \infty$.

Let Γ be the proper discrete group of isometries of H (we assume that H satisfies the axiom of visibility). Let $L(\Gamma) = \overline{\{\varphi(x),\ \varphi \in \Gamma\}}$, where $x \in H$. The set $L(\Gamma)$ is said to be *limiting*; it does not depend on the choice of the point x, lies in $H(\infty)$, is closed and invariant under Γ. We call two points $x, y \in H(\infty)$ *dual* if there exists a sequence $\varphi_n \in \Gamma$ such that for some (and consequently for any) point $z \in H$

$$\varphi_n^{-1}(z) \to x, \quad \varphi_n(z) \to y \quad \text{as} \quad n \to \infty.$$

If two points $x, y \in H(\infty)$ are dual, then $x, y \in L(\Gamma)$; for any $x \in L(\Gamma)$ there exists a dual point $y \in L(\Gamma)$ (possibly $x = y$). $L(\Gamma)$ can be classified by the action of Γ on H.

***Theorem* 4.11.** *Let* Γ *be the proper discrete group of isometries of a manifold* H *satisfying the axiom of visibility. Then one of the following three possibilities holds*:

1) $L(\Gamma) = \{x\}$; *in this case every isometry* $\varphi \in \Gamma$ *is parabolic, and* x *is a common fixed point.*

2) $L(\Gamma) = \{x, y\}$; *in this case every isometry* $\varphi \in \Gamma$ *is axial,* x *and* y *are common fixed points, and* Γ *is infinite cyclic*;

3) $L(\Gamma)$ *is an infinite set*; *in this case* Γ *does not have common fixed points*; *any two points* $x, y \in L(\Gamma)$ *are dual*; *if* $x \neq y$, *then there exist open sets* $U \ni x$, *and* $V \ni y$ *and points* $x' \in U$, $y' \in V$ *that are fixed points of some axial isometry* $\varphi \in \Gamma$.

Remark 4.1. Many of the theorems above (or many of their analogues) can be proved without the assumption that the curvature is non-negative (but that the axiom of visibility holds, see [50], §2). The results obtained may be generalized to manifolds without focal points satisfying the axiom of visibility (because the arguments use either the "law of cosines", for which the axiom of visibility serves as a substitute, or certain properties of convex functions, which hold for manifolds both of non-positive curvature and without focal points, see §1.8). The results above can be used for the investigation of fundamental groups, ends, and convex functions on an Hadamard manifold H (see [106], §§9–11).

§5. Limiting spheres

1. Let H be a simply-connected Riemannian manifold without conjugate points. A *limiting sphere* in H is a hypersurface orthogonal to a bundle of asymptotic geodesics.(1) It is given by a point $p \in H(\infty)$ (corresponding to the asymptotic bundle) and a point $x \in H$, or by one vector $v \in SH$ (such that $x = \pi(v)$ and $p = \gamma_v(+\infty)$). We call the limiting sphere $L(x, p)$ or $L(v)$, respectively.

In the simplest case of a Lobachevskii plane in the Poincaré model, $L(x, p)$ is a neighbourhood (punctured at the point p) touching the absolute (that is, the unit circle in $\mathbf{R}^2$) at p and passing through the point x on the unit circle.(2)

For more general classes of Riemannian metrics there exist various approaches to the construction of limiting spheres. For one of these, which goes back to Busemann [13], the limiting sphere is defined as a level surface of the *Busemann function* f_v: $H \to \mathbf{R}$ given by the equation

$$f_v(x) = \lim_{t\to\infty} (\rho(x, \gamma_v(t)) - t). \tag{5.1}$$

This approach is developed systematically in [51] and [106] for manifolds of non-positive curvature. The existence of the limit in (5.1) and the properties of limiting spheres are established by means of the law of cosines. Some generalizations of these results were obtained by Eschenburg in [60].

In another approach due to Grant [65], Hedlund [72], and partially Busemann [13], the limiting sphere $L(v)$ is constructed as the limit of the spheres $S^{p-1}(\gamma_v(t), t)$, as $t \to \infty$. This method, developed in [34] and [35], gives more information (in any event for complex manifolds); it leads to a construction of limiting spheres on manifolds satisfying the axiom of asymptoticity; also to a proof that $L(v)$ is a submanifold of H of class C^{r-2} (where r is the smoothness class of the Riemannian metric) and to a number of other important properties.

The "equipping" of the limiting spheres $L(v)$, $v \in SM$, with orthornormal vectors (that is, the union of sets like $\{w \in SM : \gamma_v(+\infty) = \gamma_w(+\infty), \pi(w) \in L(v)\}$, forms a continuous fibration of SH that is invariant under the geodesic flow.(3) This circumstance opens up a further possibility for the construction of limiting spheres: these arise as projections into H of fibres of the fibrations in SH that are invariant under the geodesic flow. This approach is possible for manifolds of negative curvature: in this case the compressing and expanding fibrations W^- and W^+ are invariant (see §1.9).

(1) Usually limiting spheres are called horospheres, but we reserve this term for other purposes (see §5.2).

(2) It is useful to present at the same time the contrasting case of the plane $\mathbf{R}^2$ in which the limiting sphere $L(x, p)$ is the straight line through x orthogonal to the direction corresponding to p.

(3) Depending the choice of direction of the equipping vectors "within" or "without" the limiting sphere we obtain two fibrations; the first tangent to the distribution X^-, and the second to X^+.

We now proceed to the exact statements, following the scheme proposed in [34] and [35].

2. Invariant fibrations for a geodesic flow.

***Theorem* 5.1** (see [34], Theorems 6.1 and 6.3, [35], Lemma 2). *Let H be a simply-connected Riemannian manifold with a compact factor and a Riemannian metric of class C^r, $r \geqslant 3$, without conjugate points and satisfying the axiom of asymptoticity. Then the distributions X^- and X^+ are integrable, and their maximal integral submanifolds form continuous C^{r-2}-fibrations $\mathfrak{S}^-$ and $\mathfrak{S}^+$, respectively, (that is, $\mathfrak{S}^-(v)$- and $\mathfrak{S}^+(v)$-submanifolds in SH of class C^{r-2}), which are invariant under the geodesic flow f^t.*

$\mathfrak{S}^-(v)$ and $\mathfrak{S}^+(v)$ are said to be the *stable* and *unstable horosphere* passing through the linear element v.

Some properties of horospheres are established in the following theorems (we cite them only for $\mathfrak{S}^-(v)$; for $\mathfrak{S}^+(v)$ analogous theorems hold).

***Theorem* 5.2** (see [34], Propositions 6.1 and 6.2). *Under the conditions of Theorem* 5.1:

1) *$\mathfrak{S}^-(v)$ is a connected $(p-1)$-dimensional ($p = \dim M$), closed submanifold of SH;*

2) $\mathfrak{S}^-(-v) = \mathfrak{S}^+(v)$.

3) *if φ is an isometry of H, then $d\varphi\mathfrak{S}^-(v) = \mathfrak{S}^-(d\varphi v)$ (see* 2.4).

4) *there exists a $\delta > 0$ such that for any $v \in SH$ and $w \in \mathfrak{S}(v) \cap B(v, \delta)$ (where $B(v, \delta)$ is the ball in SH with centre at v and of radius δ) the geodesic $\gamma_w(t)$ is orthogonal to the submanifold $\pi(\mathfrak{S}^-(v) \cap B(v, \delta))$ and is asymptotic to the geodesic $\gamma_v(t)$;*

5) $f^t(\mathfrak{S}^-(v)) \cap \mathfrak{S}^-(v) = \varnothing$ *for any $t \neq 0$ (a sharper result is given in Proposition* 5.1).

We denote by $\mathfrak{S}^0$ the smooth fibration of SH formed by the trajectories of the geodesic flow.

***Theorem* 5.3** (see [34], Theorem 6.4). *Under the conditions of Theorem* 5.1 *$\mathfrak{S}^-$ and $\mathfrak{S}^0$ are integrable in the sense of* [3] *(see* §4), *and the fibres of the corresponding fibration (denoted by $\mathfrak{S}^{-0}$) are integral submanifolds of the distribution $X^- \oplus Z$ (Z is the one-dimensional distribution generated by the vector field that specifies the flow). $\mathfrak{S}^{-0}$ is invariant under the geodesic flow, and its fibres have the following properties:*

1) *$w \in \mathfrak{S}^{-0}(v)$ if and only if the geodesics $\gamma_v(t)$ and $\gamma_w(t)$ are positively asymptotic;*

2) *if φ is an isometry of H, then $d\varphi(\mathfrak{S}^{-0}(v)) = \mathfrak{S}^{-0}(d\varphi v)$.*

$\mathfrak{S}^{+0}$ can be constructed similarly. $\mathfrak{S}^{-0}(v)$ and $\mathfrak{S}^{+0}(v)$ are called a *stable* and *unstable sheet* passing through v.

Let M be a compact Riemannian manifold satisfying the axiom of asymptoticity. Theorem 5.2, 3) and Theorem 5.3, 2) allow us to "lower" horospheres and sheets to SM. In particular, horospheres and sheets can be constructed on any compact Riemannian manifold satisfying one of the following conditions:

1) the manifold does not have focal points;

2) the manifold does not have conjugate points and satisfies the axiom of visibility;

3) the manifold does not have conjugate points, $d(M) = 2$, and $\chi(M) < 0$ (see Theorem 3.5, 3)).

Let $v \in SH$, $w \in \mathfrak{S}^-(v)$. We put

$$\Psi(t) = \rho_{\mathfrak{S}^-(f^t(v))}(f^t(v),\ f^t(w)).$$

$\mathfrak{S}^-$ is said to be contracting as $t \to +\infty$ if $\Psi(t) \to 0$ as $t \to +\infty$. Similar definitions (with t replaced by $-t$) hold for $\mathfrak{S}^+$. We mention that invariant fibrations for a Y-flow (in particular, for a geodesic flow on a compact Riemannian manifold of negative curvature) have these properties (see 1.9).

***Theorem* 5.4** (see [34], Theorems 7.3 and 7.4). *Let M be a compact Riemannian manifold without focal points and let v be an element of uniqueness. Then $\Psi(t) \to +\infty$ monotonically, as $t \to -\infty$.*

2) *Let M be a compact Riemannian manifold without focal points satisfying the axiom of uniform visibility. Then $\mathfrak{S}^-$ is expanding, as $t \to -\infty$. Moreover, if* $\dim M = 2$, *then $\mathfrak{S}^-$ is compressing, as $t \to +\infty$.*

The following two propositions give sufficient conditions for each $\mathfrak{S}^-(v)$ to be dense in SM.

***Theorem* 5.5** (see [52], Theorem 6.1 and [55], §4). 1) *Let M be a complete Riemannian manifold of non-negative curvature satisfying the axiom of visibility. Then $\overline{\mathfrak{S}^-(v)} = SM$ for each $v \in SM$ if and only if M is compact.*

2) *Let M be a compact two-dimensional Riemannian manifold without conjugate points and $\chi(M) < 0$. Then $\mathfrak{S}^-(v) = SM$ for each $v \in AM$.*

In conclusion we mention one consequence of Theorem 5.1.

***Theorem* 5.6** (see [34], Theorem 7.1). *Under the conditions of Theorem* 5.1, *the distributions X^- and X^+ are continuous.*

Special cases of this assertion are in [53] (see Proposition 2.13).

3. Limit spheres.

Here we assume that H is a simply-connected Riemannian manifold with a compact factor and a Riemannian metric without conjugate points satisfying the axiom of asymptoticity. Let $v \in SH$, $x = \pi(v)$, $q = \gamma_v(+\infty)$. The set $L(x,\ q) = \pi(\mathfrak{S}^-(v))$ is called the limiting sphere with centre at q passing through x. The following theorem describes the basic properties of limiting spheres on manifolds without conjugate points (some of these were mentioned in §5.1).

Theorem 5.7 (see [34], Theorem 7.2 and Lemma 6.4, and [35], Lemma 2).

1) *For any $x \in H$ and $q \in H(\infty)$ there exists a unique limiting sphere with centre at q passing through x.*

2) *The limiting sphere $L(x, q)$ is a $(p-1)$-dimensional closed submanifold of H of class C^{r-2} (where r is of the smoothness class of the Riemannian metric and $\gamma = \dim M$).*

3) *For any $v \in SH$ for which $\gamma_w(+\infty) = q$ the geodesic $\gamma_w(t)$ intersects the limiting sphere $L(x, q)$ orthogonally at a unique point.*

4) *For any $y \in L(x, q)$ there exists a sequence of numbers $t_n \to +\infty$ and points $y_n \to y$ such that $y_n \in S^{p-1}(\gamma_v(t_n), t_n)$. (Moreover, any compact submanifold of $L(x, q)$ can be approximated by a sequence of compact submanifolds of $S^{p-1}(\gamma_v(t_n), t_n)$.)*

5) *If φ is an isometry on H, then $\varphi(L(x, q)) = L(\varphi(x), \varphi(q))$ (the action of φ is extended to $\bar{H}$ in accordance with (4.4)).*

6) *$\mathfrak{S}^-(v)$ is the equipping of $L(x, q)$ with orthogonal unit vectors directed towards the same side as v. $\mathfrak{S}^+(v)$ is the equipping of the limiting sphere $L(x, q')$ ($q' = \gamma_v(-\infty)$) with orthogonal unit vectors directed towards the same side as $-v$.*

For $y \in L(x, q)$ we denote by $\gamma(y, t)$ the geodesic passing through y that is orthogonal to the limiting sphere $L(x, q)$ and is parametrized in such a way that $\gamma(y, 0) = y$ and $\gamma(y, +\infty) = q$. The following proposition means that limiting spheres with a common centre are parallel.

Proposition 5.1 (see [34], Proposition 7.1). *Let $y \in H$ and t_0 be such that $\gamma(x, t_0) \in L(y, q)$. Then*

$$\rho(L(x, q), L(y, q)) = \rho(x, \gamma(x, t_0)) = |t_0|.$$

4. The horocycle topology.

For $x \in H$ and $q \in H(\infty)$ we put

$$B^-(x, q) = \bigcup_{y \in L(x, q)} \bigcup_{t>0} \gamma(y, t), \quad B^+(x, q) = \bigcup_{y \in L(x, q)} \bigcup_{t<0} \gamma(y, t).$$

The set $B^-(x, q)$ is called the interior of the limiting sphere or the (open) limiting ball with centre at q and passing through x. The set $B^+(x, q)$ is called the exterior of the limiting sphere.

Proposition 5.2 (see [34], Proposition 7.2).

1) $H = B^-(x, q) \cup B^+(x, q) \cup L(x, q)$.

2) *The sets $B^-(x, q)$ and $B^+(x, q)$ are open and simply-connected.*

3) $\partial B^-(x, q) = L(x, q)$.

4) $B^-(x, q) = \bigcup_{t>0} B^p(\gamma(t), t)$, *where $B^p(y, r)$ is the ball in H with centre at y and radius r; $\gamma(t)$ is a geodesic for which $\gamma(0) = x$ and $\gamma(+\infty) = q$.*

We put $\bar{B}^-(x, p) = B^-(x, p) \cup \{p\}$. The sets $\bar{B}^-(x, p)$, called closed limiting balls, constructed from all $x \in H$ and $p \in H(\infty)$ give a topology h in H, which is called *horocyclic.*

***Proposition* 5.3.** 1) *The topology h has the properties formulated in Theorem* 4.6, 1), 2), *and* 5) (*and are admissible in the sense of* [106], p.50).

2) *The topology h is weaker than* τ.

3) *For an Hadamard manifold H satisfying the axiom of visibility the topologies and h are* τ *equivalent.*

The first two assertions follow directly from Theorem 5.7 and Propositions 5.1 and 5.2. In the case of the space $\mathbf{R}^n$ the topology h is not Hausdorff. 3) is proved in [106] (see Proposition 4.8).

5. Further properties of limiting spheres.
On manifolds of non-positive curvature or without focal points one can obtain a certain amount of additional information about properties of limiting spheres. Using the convexity of spheres on manifolds without focal points we can prove the following proposition.

***Proposition* 5.4** (see [34], Proposition 7.3). *If a manifold H with a compact factor does not have focal points, then the limiting sphere is convex.*

It follows from this that the function $\varphi(t) = \rho(\gamma(t), \sigma(t))$, where γ and σ are positively asymptotic geodesics, decreases monotonically for $t > 0$. The following fact was established in [103] (see Proposition 4) without the assumption that H has a compact factor. From Theorem 5.4, 2) it follows also that when $\dim H = 2$ and H satisfies the axiom of uniform visibility, $\varphi(t) \to 0$, as $t \to +\infty$.

In [75] Heintze and Im Hof, using comparison theorems for Jacobi fields (see Theorem 2.3), obtained some results on properties of horospheres on an Hadamard manifold H whose curvatures $K_x(P)$ satisfy the conditions

$$-b^2 \leqslant K_x(P) \leqslant -a^2, \quad a \in [0, \infty), \quad b \in [0, \infty].$$

For example, in [75] (see Theorem 4.6) it is proved that for any limiting sphere $L(x, q)$ and for any $y, z \in L(x, q)$

$$\frac{2}{a} \sinh \frac{a}{2} \rho(y, z) \leqslant \rho_{L(x, q)}(y, z) \leqslant \frac{2}{b} \sinh \frac{b}{2} \rho(y, z).$$

We cite one further result (see [75], Theorem 4.7). Let $L(x, q)$ be a limiting sphere, γ a geodesic touching $L(x, q)$ at some point, and σ the projection of[(1)] γ onto $L(x, q)$. y, z are the end points of σ. Then

$$\frac{2}{b} \leqslant \rho_{L(x, q)}(y, z) \leqslant \text{length of } \sigma \leqslant \frac{2}{a}.$$

[(1)] That is, $\sigma(t)$ is the point of intersection of the limiting sphere $L(x, q)$ with the geodesic joining $\gamma(t)$ to q.

Im Hof obtained in [79] interesting results on the geometry of limiting spheres. In particular, he proved that the family of limiting spheres passing through two given points in H cut off in $H(\infty)$ a set homeomorphic to S^{p-2} (so that when $p = 2$, there exist exactly two limiting spheres passing through two given points—a result obtained earlier in [65]).

6. The Busemann function.

Let H be an Hadamard manifold, $x \in H$, and $\gamma_v(t)$ any geodesic in H with initial vector v. The function $t \to p(x, \gamma_v(t)) - t$ decreases monotonically and is bounded below. Therefore, (5.1) gives the well-defined function on H, which is called the *Busemann function*. It has the following properties:

1) f_v is uniformly continuous and convex; moreover

$$| f_v(x) - f_v(y) | \leqslant \rho(x, y);$$

2) if two geodesics γ_v and γ_w are asymptotic, then $f_v - f_w$ is constant.

***Proposition* 5.5** (see [106], §3 and [35], §6). *$L(x, q) = \{y \in H: f_v(y) = f_v(x)\}$, where v is a vector in SH such that $\gamma_v(0) = x$ and $\gamma_v(+\infty) = q$.*

This allows us to regard the Busemann function as a function of two variables: points $x \in H$ and points $q = \gamma_v(+\infty) \in H(\infty)$. The function $f_q(x)$ is jointly continuous in the variables, and as Eberlein has shown (unpublished; for published proofs, see [75] and [60]) it is twice continuously differentiable in x.

We fix $q \in H(\infty)$ and consider the vector field V_q in H that is given by

$$V_q(x) = \dot{\gamma}_{xq}(0)$$

and is called the radial vector field. It is orthogonal to the limiting spheres $L(x, q)$, moreover, grad $f_q = -V_q$ (see [106], Proposition 3.5).

(5.1) specifies the Busemann function on an arbitrary simply-connected Riemannian manifold H without conjugate points, therefore, it defines the limiting spheres on H as level surfaces of f_v. Some properties of limiting spheres can be established under fairly weak restrictions. For example, a property analogous to parallelism can be proved provided that the distribution X is continuous (see [60], Proposition 3). However, to prove the properties of limiting spheres expressed in Theorem 5.7 more rigid restrictions are needed. The most general conditions of this kind are given in Theorem 5.1. In [60] Eschenburg establishes some of these properties under a stronger assumption, the so-called condition of restricted asymptoticity (which means that $\| Y_\xi(t) \| \leqslant$ const. uniformly in all $t \geqslant 0$, $v \in SH$, and $\xi \in X^-(v)$, $\|\xi\| = 1$; this condition is satisfied by manifolds without focal points and manifolds of Anosov type). With the same assumption he proves that the Busemann function $f_q(x)$ lies in C^2. From Theorem 5.7, 2) there follows a stronger assertion.

Proposition **5.6** (see [35], §6). *On a simply-connected Riemannian manifold with a compact factor satisfying the axiom of asymptoticity, the Busemann function $f_q(x)$ lies in C^{r-2} (where $r \geqslant 3$ is the smoothness class of the Riemannian metric).*

§6. Topological properties of geodesic flows

1. Topological transitivity and mixing.

The most general results establishing topological transitivity and mixing are due to Eberlein. We denote by Ω a set of non-wandering points for a geodesic flow in *SM.*

Theorem **6.1** (see [50], Theorem 3.7). *Let M be a complete Riemannian manifold satisfying the axiom of uniform visibility, and let $\Omega = SM$. Then the geodesic flow is topologically transitive on SM. In particular a geodesic flow on a compact Riemannian manifold satisfying the axiom of visibility is topologically transitive.*

For two-dimensional compact manifolds without conjugate points with $\chi(M) < 0$ (that is, satisfying the axiom of uniform visibility, see Theorem 4.4) this was proved by Green in [67].

Let M be a complete Riemannian manifold, H its universal Riemannian covering manifold such that $M = H/\Gamma$, where Γ is the proper discrete group of isometries on H (see §4.6). The topological transitivity of a geodesic flow is closely connected with the action of Γ on $H(\infty)$. By reproducing the proof of Theorem 4.14 in [52] (see the implication (1) $\Rightarrow$ (2)) we can prove the following proposition.

Proposition **6.1**. *Under the conditions of Theorem 6.1, the action of Γ on $H(\infty)$ is topologically transitive.*

We look now at manifolds of non-positive curvature.

Theorem **6.2** (see [52], Theorem 4.14). *Let $M = H/\Gamma$ be a complete Riemannian manifold of non-positive curvature and $\Omega = SM$. Then the following conditions are equivalent:*

1) *a geodesic flow is topologically transitive on SM;*

2) *the action of Γ on $H(\infty)$ is topologically transitive;*

3) $\overline{\{\varphi(x)\colon \varphi \in \Gamma\}} = H(\infty)$ *for each* $x \in H(\infty)$;

4) $\overline{\mathfrak{S}^{-0}(v)} = SM$ *for some* $v \in SM$;

5) $\overline{\mathfrak{S}^{-0}(v)} = SM$ *for each* $v \in SM$;

6) *for any two open sets U and V there exists a sequence of numbers $t_n \to +\infty$ such that $f^{t_n}(U) \cap V \neq \varnothing$.*

***Theorem* 6.3.** *Let M be a compact Riemannian manifold of non-positive curvature satisfying the axiom of visibility. Then*

1) *a geodesic flow has the properties* 1)–6) *of Theorem* 6.2;

2) $\overline{\mathfrak{S}^-(v)} = SM$ *and* $\overline{\mathfrak{S}^+(v)} = SM$ *for each* $v \in SM$ (*see Theorem* 5.5);

3) *a geodesic flow mixes topologically* (*see* [52], *Theorem* 6.3).

The last assertion is true for any (not necessarily compact) manifolds satisfying the conditions of Theorem 6.2 and the axiom of visibility.

2. Closed geodesics.(1)

In the case of manifolds of negative curvature we can gain fuller information on the topological properties of geodesic flows, using Theorem 3.1. Thus, their topological transitivity and topological mixing are consequences of the corresponding results for the theory of Y-flows. We now consider the question of periodic trajectories of geodesic flows, to which there correspond closed geodesics on the manifold.

***Theorem* 6.4** (see [3], Theorem 3). *Let M be a compact Riemannian manifold of negative curvature. Then a geodesic flow has a countable dense set of periodic trajectories.*

Thus, on a compact manifold of negative curvature closed geodesics form a countable dense set (in SM). Also the number of closed geodesics of length not exceeding R is finite. We denote this number by $v(R)$.

The first asymptotic estimate of the number of closed geodesics on a compact Riemannian manifold of negative curvature was given by Sinai in [38]. His result was substantially refined by Margulis.

***Theorem* 6.5** (see [26], [28]). *There exists a* $d > 1$ *such that*

$$\lim_{R\to\infty} \frac{dR\nu(R)}{e^{dR}} = 1. \tag{6.1}$$

The number d is the topological entropy of the geodesic flow, and is the coefficient of expansion of the sets lying on $\mathfrak{S}^+(v)$ relative to a certain special measure constructed by Margulis (see [27], §6, and also Footnote 2 to Proposition 12.1). In [26] there are also other asymptotic characteristics of manifolds of negative curvature.

***Theorem* 6.6.** *Let M be a two-dimensional compact Riemannian manifold without focal points and with* $\chi(M) < 0$. *Then a geodesic flow has a countable dense set of periodic trajectories, that is, the closed geodesics form a countable dense set* (*in SM*).

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(1)We do not dwell on results about the existence and the number of closed geodesics for arbitrary compact Riemannian manifolds (in this connection, see [17], §7.5, Remark 6; also [21] and [7]), but we confine ourselves to manifolds of negative curvature and some other special cases.

This theorem follows from Theorem 7.5 and from the results of [80]. We can also obtain an asymptotic estimate of the number of closed geodesics, namely,

$$\nu(R) \geqslant ce^{h(f)R}, \tag{6.2}$$

where $c > 0$ is a certain constant and $h(f)$ is the topological entropy of the flow f^t. Theorem 6.6 and the estimate (6.2) (with $h(f)$ replaced by $h_\mu(f)$) are valid in the multi-dimensional case if we assume in addition that the manifold M satisfies the axiom of uniform visibility and that $\mu(\Lambda_0) > 0$ (see [80], Theorem 4.3; the set Λ_0 is defined in §3.2).

We quote one more lower estimate of the topological entropy of a geodesic flow, which is due Manning [88].

***Theorem* 6.7.** *Let M be a compact Riemannian manifold, H its universal Riemannian covering manifold, $B(x, r)$ the ball in H with centre x and radius r, $V(x, r)$ the volume of $B(x, r)$.*

1) *The following limit exists*:

$$\lim_{r\to\infty} \frac{\log V(x, r)}{r} = \lambda \geqslant 0;$$

2) $h(f) \geqslant \lambda$;

3) *if the curvature of M is non-positive, then $h(f) = \lambda$.*

§7. Ergodic properties of geodesic flows

1. Isomorphism to a Bernoulli flow.

According to Theorem 3.1, a geodesic flow on a compact Riemannian manifold of negative curvature is a Y-flow. This permits us to apply to the study of its ergodic properties, relative to the measure μ (see § 1.2) the general results of the theory of Y-flows due to Anosov [2], Sinai [37] and Bunimovich [14]; and to prove the following theorem.

***Theorem* 7.1.** *A geodesic flow on a compact Riemannian manifold of negative curvature is isomorphic to a Bernoulli flow; in particular, it is ergodic, has the mixing property of all degrees and the K-property.*

We leave aside such properties of geodesic flows on manifolds of negative curvature as the existence of Markov partitions, Gibbs measures, the central limit theorem and the like, which have been established within the framework of the general theory of Y-systems (and even systems satisfying axiom A of Smale) by means of the construction of symbolic models for them. The proofs of the corresponding assertions contain nothing specifically geometric, althouth historically symbolic models emerged precisely for geodesic flows.

Geodesic flows on manifolds without conjugate or focal points have, generally speaking, fairly weak hyperbolic properties (see §3.2) and the investigation of their ergodic properties can be pursued by using the results of the general theory of dynamical systems with non-zero characteristic Lyapunov indicators (see [32]-[34]). Here the central question is whether the set Λ defined by (1.10) has positive measure. In general the answer is not known, but the following alternative holds.

Theorem 7.2 (see [34], Theorem 9.1). *Let M be a compact Riemannian manifold without focal points satisfying the axiom of uniform visibility. Then either $\mu(\Lambda) = 0$ or $\mu(\Lambda) = 1$. In the latter case the geodesic flow is isomorphic to a Bernoulli flow.*

Another version of this alternative is as follows.

Theorem 7.3 (see [34], Theorem 9.5). *Let M be a compact Riemannian manifold without conjugate points satisfying the strong axiom of uniform visibility. Then either $\mu(\Lambda) = 0$ or $\mu(\Lambda) = 1$. In the latter case the geodesic flow is isomorphic to a Bernoulli flow.*

More complete and definitive results can be obtained in the two-dimensional case.

Theorem 7.4 (see [34], Theorem 9.2). *Let M be a compact two-dimensional Riemannian manifold without conjugate points, with $\chi(M) < 0$. Then either $\mu(\Lambda) = 0$ or $\mu(\Lambda) = 1$. In the latter case the geodesic flow is isomorphic to a Bernoulli flow.*

This and Theorem 3.4 leads to the following theorem.

Theorem 7.5 (see [34], Theorems 9.3 and 9.5). *A geodesic flow on a compact two-dimensional Riemannian manifold with $\chi(M) < 0$ is isomorphic to a Bernoulli flow if one of the following conditions holds:*

1) *the Riemannian manifold does not have focal points;*
2) *the entropy of the geodesic flow is positive.*

A weaker proposition is in [23].

We quote one other condition, which is weaker than 1) of 7.5, but still guarantees that a geodesic flow is isomorphic to a Bernoulli flow. We put $A = \{v \in SM$: no two points on the geodesic $\gamma_v(t)$ are focal$\}$.

Theorem 7.6 (see [32], Theorem 10.8). *Let M be a compact two-dimensional Riemannian manifold without conjugate points and with $\chi(M) < 0$. Further, let $\mu(A \cap \Lambda) > 0$. Then the geodesic flow is isomorphic to a Bernoulli flow. Moreover, $\mu(\Lambda) = \mu(A) = 1$.*

2. **Formulae for the entropy of a geodesic flow.**

Our presentation here follows [35]. For $v \in SM$ we define a linear map S_v: $v^\perp \to v^\perp$ by

$$S_v w = K\xi(w)$$

(we recall that $v^{\perp}$ is the orthogonal complement to v in $T_{\pi(v)}M$, and $\xi(w)$ is a vector in $X^{-}(v)$ such that $d\pi\xi(w) = w$). S_v is a linear bounded self-adjoint operator. We denote by $\{e_i\}$, $(i = 1, ..., p-1, p = \dim M)$, the orthonormal basis in $v^{\perp}$ consisting of the eigenvectors of S_v; and by $K_i(v)$ $(i = 1, ..., p-1)$ the corresponding eigenvalues. The $K_i(v)$ are called the principal curvatures of the element v, and the vectors $e_i(v)$ the directions of the principal curvatures. The justification of these names is the following assertion.

***Theorem* 7.7** (see [35], Lemma 3). *Let M be a compact manifold with a Riemannian metric of class C^4 satisfying the axiom of asymptoticity. Then S_v for any $v \in SM$ is the operator of the second quadratic form of the limiting sphere $L(\pi(v), \gamma_v(+\infty))$ at the point $\pi(v)$.*

For $v \in SM$ we put

$$K_{ij}(v) = \langle R(v, e_i(v))v, e_j(v)\rangle, \quad \bar{K}_i(v) = K_{ii}(v).$$

The last number is the curvature of M in the two-dimensional direction given by the vectors v and $e_i(v)$. We now quote the two basic formulae for the entropy of a geodesic flow.

***Theorem* 7.8** (see [35]). *The entropy of geodesic flow on a compact Riemannian manifold without conjugate points the following equation holds*:

$$h_{\mu}(f^1) = -\int_{SM} \mathrm{Sp}\,(S_v)\,d\mu(v) = -\int_{SM} \sum_{i=1}^{p-1} K_i(v)\,d\mu(v). \tag{7.1}$$

($\mathrm{Sp}(S_v)$ *is the trace of* S_v),

$$h_{\mu}(f^1) = -\int_{SM} \sum_{i=1}^{p-1} \frac{K_i(v)}{1+K_i^2(v)}\,(1-\bar{K}_i(v))\,d\mu(v).$$

In the two-dimensional case (7.1) can be rewritten in a somewhat different form.

***Theorem* 7.9** (see [35], Corollary 7). *Let M be a compact two-dimensional manifold without conjugate points. Then*

$$h_{\mu}(f^1) = -\int_{SM} \tan\alpha(v)\,d\mu(v),$$

where $\alpha(v)$ is the angle between the lines $X^{-}(v)$ and $(V_{\Gamma})_v$ (see §1.2) counted off from the last anti-clockwise arrow.

An important consequence of Theorem 7.8 is the following assertion, which supplements Theorem 7.2.

***Proposition* 7.1** (see [35], Corollaries 4 and 5). *Let M be a compact Riemannian manifold without focal points. We suppose that there exists a $v_0 \in SM$ for which $K_i(v_0) < 0$ for all $i = 1, ..., p-1$ (that is, the limiting sphere $L(\pi(v_0), \gamma_{v_0}(+\infty))$ is strictly convex at $\pi(v_0)$): Then $\mu(\Lambda) > 0$. In* *particular, the entropy of the geodesic flow is positive, and if the manifold satisfies the axiom of uniform visibility, then the geodesic flow is isomorphic to a Bernoulli flow.*

We give a further consequence of 7.8.

Proposition 7.2 (see [35], Corollary 3). *A geodesic flow on a non-plane Riemannian manifold without focal points has positive entropy.* (A Riemannian manifold is said to be plane if its curvature tensor is identically zero.)

To conclude this subsection we indicate two identities for the curvatures $K_i(v)$ and $\overline{K}_i(v)$.

Proposition 7.3 (see [35], Corollaries 2 and 6). *Let M be a compact Riemannian manifold without conjugate points. The following relations hold*:

$$\int_{SM} \sum_{i=1}^{p-1} K_i(v) \frac{K_i^2(v)+\overline{K}_i(v)}{K_i^2(v)+1}\, d\mu(v) = 0,$$

$$\int_{SM} \sum_{i=1}^{p-1} K_i^2(v)\, d\mu(v) = -\frac{\omega_{p-1}}{p} \int_M K(x)\, d\nu(x), \tag{7.2}$$

where ω_{p-1} is the volume of the $(p-1)$-dimensional unit sphere in $\mathbf{R}^p$, $K(x)$ is the scalar curvature at $x \in M$ (if $\{e_i\}_{i=1}^p$ is an orthonormal basis at x, then $K(x) = \sum_{i,j=1}^{p} K_x(P_{i,j})$ where $P_{i,j}$ is the plane spanned by the vectors e_i and e_j: see [17], p.114, *and also* [66]).

3. Estimate of the entropy of a geodesic flow.
From (7.1) and (7.2) and the Cauchy-Bunyakovskii inequality one can derive the following upper bound on the entropy.

Theorem 7.10. *Let M be a p-dimensional compact Riemannian manifold without conjugate points. Then*[1]

$$h_\mu(f^1) \leqslant \omega_{p-1}\nu(M) \sqrt{\frac{p-1}{p} \frac{1}{\nu(M)} \int_M (-K(x))\, d\nu(x)}\ . \tag{7.3}$$

For two-dimensional manifolds without focal points this result was obtained by Manning in [87] (see Theorem 1). In the same paper he obtained a lower bound on the entropy.

[1]Note that, as follows from the results of [66] (see also (7.2)), $\int_M K(x)\, d\nu(x) \leqslant 0$ if the Riemannian metric in M does not have conjugate points.

***Theorem* 7.11** (see [87], Theorem 2). *Let M be a compact surface with a Riemannian metric of non-positive curvature of class C^3. Then*

$$h_\mu(f^1) \geqslant 2\pi\nu(M) \int_M \sqrt{-K_0(x)}\, d\nu(x),$$

where $K_0(x)$ is the curvature[(1)] *of the surface at x.*

For compact surfaces with an arbitrary Riemannian metric σ and Euler characteristic $\chi(M) = 0$ Katok obtained in [81] interesting relations between the asymptotic number of closed geodesics

$$P_\sigma = \varlimsup_{R\to\infty} \frac{\log \nu_\sigma(R)}{R}$$

(where $\nu_\sigma(R)$ is the number of closed geodesics of length $\leqslant R$), the topological entropy $h^\sigma(f^t)$, and the metric entropy $h^\sigma_{\mu(\sigma)}(f^t)$ (where $\mu(\sigma)$ is the measure on SM induced by the Riemannian metric σ). Namely,

$$P_\sigma \geqslant h^\sigma(f^t) \geqslant \left(-\frac{2\pi E}{V_G}\right)^{1/2},$$

where V_σ is the volume of M in the metric σ; if σ does not have focal points, then

$$h^\sigma_{\mu(\sigma)}(f^t) \leqslant \rho_\sigma \left(-\frac{2\pi E}{V_\sigma}\right)^{1/2} \leqslant \left(-\frac{2\pi E}{V_\sigma}\right)^{1/2},$$

where

$$\rho_\sigma = \int_M \rho_\sigma(x)^{1/2} d\mu(\sigma_0),$$

and the function $\rho_\sigma(x)$ is uniquely determined by the condition $\sigma(x) = \rho_\sigma(x)\sigma_0(x)$ (where $\sigma_0(x)$ is the metric of constant negative curvature, see [107]); compare with (7.3) and (6.2)

§8. Geodesic flows on manifolds of Anosov type

1. Manifolds of Anosov type.

A compact Riemannian manifold M is said to be of Anosov type if there is a Riemannian metric on M (which in this subsection is denoted by $\langle\,,\rangle^*$) in which the geodesic flow is a Y-flow.

Hopf [77] already drew attention to the fact that geodesic flows on manifolds of negative curvature and on manifolds having only small parts of positive curvature may have similar properties. What is decisive here is not that the curvature is negative but that the trajectories have a certain instability, and this can occur even if there are parts with positive curvature. Hopf had in mind the case when focal points are altogether absent. It is substantially more complicated to construct a Riemannian metric with focal points such that the geodesic flow corresponding to it is a Y-flow. Examples of this kind were constructed by Anosov and Gulliver [70].

(1) Note that $K(x) = 2K_0(x)$.

In [82] Klingenberg studied the geometric properties of manifolds of Anosov type with regard to a Riemannian metric $\langle , \rangle^*$ and proved the following theorem (see [82] Theorem 5.3.4).

Theorem 8.1. *Let M be a compact Riemannian manifold of Anosov type. Then the Riemannian metric* $\langle , \rangle^*$ *does not have conjugate points and satisfies the axiom of uniform visibility.*

This theorem, the general properties of Y-systems, and the results of §§6 and 7 give rise to the following properties of manifolds of Anosov type:

1) *geometric properties of Riemannian metrics*: each Riemannian metric without conjugate points satisfies the axiom of uniform visibility; the universal Riemannian covering manifold (with respect to any Riemannian metric without conjugate points) allows the compactification described in §4.4;

2) *topological properties of the same manifold*: the universal Riemannian covering manifold is homeomorphic to $\mathbf{R}^p$ ($p = \dim M$); the fundamental group $\pi_1(M)$ has exponential growth; each non-trivial Abelian sub-group of $\pi_1(M)$ is infinite cyclic;

3) *ergodic and topological properties of a geodesic flow* f^t *(with respect to the Riemann metric* $\langle , \rangle^*$*)*: the flow f^t is ergodic (and isomorphic to a Bernoulli flow); f^t mixes topologically and has a countable dense set of periodic trajectories, and their number of period $\leqslant T$ grows exponentially with T (see (6.1));

4) *topological properties of a geodesic flow* f^t *(with respect to any Riemannian metric without conjugate points)*: f^t is topologically transitive.

We also note that on manifolds of Anosov (and a fortiori of hyperbolic) type an arbitrary Riemann metric has geodesics that inherit, as it were, the properties of geodesics in the metric $\langle , \rangle^*$ (see [93], [21], [104]).

To the assertions cited we add the following.

Theorem 8.2 (see [82]). *Let M be a compact Riemannian manifold of Anosov type with Riemannian metric* $\langle , \rangle^*$. *Then the index of each closed geodesic in M is zero (for the definition of index, see* [17], §4).

In [53] Eberlein investigated the infinitesimal properties of geodesics on manifolds of Anosov type (that is, properties of solutions of the variational equations for a geodesic flow in the metric $\langle , \rangle^*$).

Theorem 8.3 (see [53], Theorem 3.2, Corollary 3.4). *Under the conditions of Theorem* 8.2

1) *there exist an* $A > 0$ *and an* $s_0 > 0$ *such that for any geodesic* $\gamma(t)$ *of any orthogonal Jacobi field* $Y(t)$ *along* $\gamma(t)$ *such that* $Y(0) = 0$, *and for any numbers* $t \geqslant s \geqslant s_0$,

$$\| Y(t) \|^* \geqslant A \, \| Y(s) \|^*;$$

2) $$\int_1^\infty \frac{1}{g(t)} dt < \infty, \quad \text{где} \quad g(t) = \inf \{ \| Y(t) \|^* : Y \in J_0(\gamma), \ Y(0) = 0, \ \| Y'(0) \|^* = 1, \ t > 0 \};$$

3) *for any geodesic* γ *and any orthogonal field* $E(t)$ *along* $\gamma(t)$ *parallel along* $\gamma(t)$ *(that is,* $E'(t) = 0$ *for all* t*), there exists a* t_0 *such that* $K_{\gamma(t_0)}(P) < 0$, *where* P *is the plane spanned by the vectors* $E(t_0)$ *and* $\dot{\gamma}(t_0)$.

Theorem 8.3 remains true if the requirement of compactness of the manifold is replaced by the assumption that $K_x(P) \geqslant -k^2$, $k > 0$, for all x and p.

2. The question of when a geodesic flow is a Y-flow is of undoubted interest. Necessary and sufficient conditions for this were obtained by Eberlein in [53] and [54]. In the light of Theorem 8.1 we can immediately confine ourselves to the analysis of Riemannian metrics without conjugate points, which allows us to apply the results of §§2–4.

Theorem 8.4 (see [53], Theorem 3.2). *Let* M *be a complete Riemannian manifold with a Riemannian metric* $\langle , \rangle$ *without conjugate points. Suppose also that* $K_x(P) \geqslant -k^2$, $k > 0$, *for all* x *and* P. *Then the following propositions are equivalent:*

1) *a geodesic flow is a* Y*-flow;*

2) $X^+(v) \cap X^-(v) = 0$ *for each* $v \in SM$;

3) $T_vSM = X^+(v) \oplus X^-(v) \oplus Z(v)$ *for each* $v \in SM$ *(where* $Z(v)$ *is the one-dimensional subspace spanned by* $V(v)$*);*

4) *there is no non-zero orthogonal Jacobi field* $Y(t)$ *along a geodesic* $\gamma(t)$ *for which* $\|Y(t)\|$ *is bounded for all* $t \in \mathbf{R}$.

Theorem 8.5 (see [53], Corollary 3.3). *Under the conditions of Theorem* 8.4 *suppose that the Riemannian metric does not have focal points. Then the following conditions are equivalent:*

1) *a geodesic flow is a* Y*-flow*

2) *there is no orthogonal Jacobi field* $Y(t)$ *along any geodesic* $\gamma(t)$ *parallel along* $\gamma(t)$ *(that is,* $\|Y'(t)\| = 0$ *for all* $t \in \mathbf{R}$).

3) $$\int_1^{\infty} \frac{1}{g(t)}\, dt < \infty.$$

Theorem 8.6 (see [53], Corollary 3.6). *Under the conditions of Theorem* 8.5 *suppose that* $\dim M = 2$. *Then a geodesic flow is a* Y*-flow if and only if for each geodesic* $\gamma(t)$ *there exists a* t_0 *such that* $K_0(\gamma(t_0)) < 0$ *(where* $K_0(x)$ *is the curvature of* M *at* x*).*

Now we give sufficient conditions for a geodesic flow to be Y-flow.

Theorem 8.7 (see [53], Corollary 3.5). *Let* M *be a compact Riemannian manifold without focal points. Suppose that for any geodesic* γ *and any orthogonal and parallel field* $E(t)$ *along* γ *there exists a* t_0 *such that* $K_{\gamma(t_0)}(P) < 0$, *where* P *is the plane spanned by* $E(t_0)$ *and* $\dot{\gamma}(t_0)$. *Then a geodesic flow is a* Y*-flow.*

In the next theorem M is a compact Riemannian manifold without focal points, H is its universal covering manifold; P, $H \to \gamma$ is the orthogonal projection of H onto the geodesic γ in H (see Proposition 1.5), and $Y(t)$ is an orthogonal Jacobi field along γ.

***Theorem* 8.8** (see [54]). *The following conditions are equivalent and imply that a geodesic flow is a Y-flow. Moreover if the curvature of M is non-positive, then these conditions are equivalent to the statement that a geodesic flow is Y-flow.*

1) *There exists a* $t_0 > 0$ *such that for any geodesic* γ *in H, any* $x \in H$ *for which* $\rho(x, \gamma) \geqslant t_0$, *and any* $v \in T_xH$, $v \neq 0$,

$$\| dP(v) \| < \| v \|.$$

2) *There exist an* $a > 0$ *and a* $c > 0$ *such that for any geodesic* γ *in H and for any* $x \in H$ *and* $v \in T_xH$,

$$\| dP(v) \| \leqslant ae^{-ct} \| v \|,$$

where $t = \rho(x, \gamma)$.

3) *There exists a* $t_0 > 0$ *such that for any geodesic* γ *in H and any* $t > t_0$

$$\| Y(t) \| > \| Y(0) \|,$$

where $Y(0) \neq 0$ *and* $\langle Y(0), Y'(0)\rangle = 0$.

4) *There exists an* $x \in H$, *a* $c > 0$ *and a* $t_0 > 0$ *such that for any geodesic* γ *in H for which* $\gamma(0) = x$ *and for any* $t \geqslant t_0$

$$\frac{d}{dt}(\log \| Y \|^2)(t) \geqslant c, \tag{8.1}$$

where $Y(0) = 0$ *and* $Y'(0) \neq 0$.

5) *There exists a* $c > 0$ *and a* $t > 0$ *such that for any geodesic* γ *in H and for any* $t \geqslant t_0$ (8.1) *is satisfied and either* $Y(0) = 0$ *and* $Y'(0) \neq 0$, *or* $\langle Y(0), Y'(0)\rangle \geqslant 0$.

In the two-dimensional case 5) was proved by Anosov (see [3]).

3. Speical examples of manifolds of Anosov type are manifolds of hyperbolic type which are defined as those that admit a Riemannian metric of negative curvature. Such manifolds were studied by Morse and Hedlund (see [95]), Green (see [67]), and others (see, for example [21], Ch. 5). Some results were obtained that were special cases of those mentioned in §8.1. Thus, in [95] it was proved that a geodesic flow on a compact surface without conjugate points is topologically transitive if the manifold is orientable and of genus > 1, or if the manifold is not orientable and of genus > 2 (such surfaces admit a metric of negative curvature).

Part II

FRAME FLOWS AND HOROCYCLE FLOWS

§9. Definition of a frame flow

1. Let M be a complete Riemannian manifold, $\dim M = p$. A *k-frame* on M is a pair $w = (x, \xi^k)$, where $x \in M$ and ξ^k is an ordered orthonormal frame $(\xi_1, \ldots, \xi_k)$, $\xi_i \in T_xM$. The set of all k-frames on M forms a locally trivial fibration $\pi\colon\ \Omega_k \to SM$, whose basis is the Stiefel manifold $V_{p-1,\,k-1}$ (the set of $(k-1)$-frames in T_xM orthogonal to ξ_1). We denote by $P^t(v, \eta^l)$ the parallel displacement of the frame $\eta^l = (\eta_1, \ldots, \eta_l)$ at the time t along the trajectory of the geodesic flow f^t passing through the linear element v. A *k-frame flow* on M is defined as a one-parameter group Φ^t of transformation of the Ω_k acting in the following way:

$$\Phi^t(w) = (f^t(v),\ P^t(v,\ \xi^{k-1})),$$

where $w = (x,\ \xi^k) = (w,\ \xi^{k-1})$. Sometimes instead of using the term "k-frame flow" we talk of a "frame flow". This definition is due to Arnol'd (see [5]).[(1)] Obviously a frame flow is a fibre bundle over a geodesic flow (see [12], §1), and moreover, for each k a k-frame flow is a natural factor of a $(k+1)$-frame flow.

More can be asserted: a k-frame flow is an $SO\ (k-1)$-extension of a geodesic flow.[(2)] For we can interpret a k-frame $(x,\ \xi^k) = (v,\ \xi^{k-1})$ as an isomorphism of $\mathbf{R}^{k-1}$ into the space T_xM spanned by the vectors $\xi_2, \ldots, \xi_k$, and we can define for each $g \in SO(k-1)$ its action on ξ^{k-1} by the formula

$$(R_g\xi^{k-1})(u) = \xi^{k-1}(g(u)), \quad u \in \mathbf{R}^{k-1}. \tag{9.1}$$

We now take Ω_k as a smooth principal right $SO(k)$-bundle over M by specifying the action of an element $g \in SO(k)$ on a k-frame (x, ξ^k) by (9.1) with $u \in \mathbf{R}^k$. To each element A of the Lie algebra of $SO(k)$ there corresponds a vector field $\lambda(A)$ on Ω_k whose integral curves coincide with the orbits of the one-parameter group $\exp tA$. To each vector $u \in \mathbf{R}^k$, $\|u\| = 1$, there corresponds a vector field $B(u)$ on Ω_k whose integral curve passing through the k-frame (x, ξ^k) coincides with the curve in Ω_k that is obtained as a result of a parallel translation of the k-frame ξ^k along the geodesic $\gamma^k_\xi(u)$. It is easy to see that the vector field $B(e_1)$ specifies the flow Φ^t. It can be shown

(1) Even earlier this definition was given by Hopf [44] for the case $k = 2$. However, he did not investigate 2-frame flows as such, but simply used them as an instrument to study ergodic properties of geodesic flows.

(2) If G is a compact connected Lie group (in our case, the group $SO(k-1)$), then a *G-extension* of a flow h^t on a manifold M is defined as a flow f^t on a manifold N such that N is a smooth principal right G-bundle (see [45], p.65), with a projection $\pi\colon N \to M$ and $\pi\,(f^t\,(w)) = h^t\,(\pi\,(w))$, $f^t \circ R_g\,(w) = R_g \circ f\,(w)$ for any $w \in N$, $g \in G$, $t \in \mathbf{R}$ (where R_g: $N \to N$ is the right action of G).

that the variational equation for the flow Φ^t is a special case of the Cartan structure equations and has the form (see [68]):

$$[\lambda(A),\ B(e_1)] = B(A(e_1)).$$

The compact manifolds $V_{p-1,\ k-1}$ form a fibering of Ω_k that is invariant under Φ^t, which we denote by $\tilde{\mathfrak{S}}^0$. It is smooth, and the maps $P^t(v,\ \xi^{k-1})$ are isometries of $\tilde{\mathfrak{S}}^0(v)$ onto $\tilde{\mathfrak{S}}^0(f^t(v))$.

When the curvature of M is negative, the following important assertion holds.

***Theorem* 9.1** (see [12], Theorem 6.1 and Proposition 6.2). *A k-frame flow on a compact Riemannian manifold M of negative curvature is a partially hyperbolic dynamical system. In particular it has an invariant and contracting fibration $\mathfrak{S}^c$ and an expanding $\mathfrak{S}^e$, where $\pi(\mathfrak{S}^c(w)) = \mathfrak{S}^-(\pi(w))$, $\pi(\mathfrak{S}^e(w)) = \mathfrak{S}^+(\pi(w))$, ($\mathfrak{S}^-$ and $\mathfrak{S}^+$ are invariants of the geodesic flow).*

$\mathfrak{S}^c$ and $\mathfrak{S}^e$ are absolutely continuous (for the definition, see [3] or [4]) and each of them is integrable together with $\tilde{\mathfrak{S}}^0$.

Remark 9.1. The existence of $\mathfrak{S}^c$ and $\mathfrak{S}^e$ for a frame flow is a consequence of the more general assertion concerning the following situation: f^t is a fibre bundle over g^t, having an invariant fibration W, that contracts at an exponential rate. Under these conditions there exists a contracting fibration $\widetilde{W}$ that is invariant relative to f^t and can be projected into W. This includes, of course, the case when g^t is a Y-flow, but this situation permits us also to investigate the case when g^t has almost everywhere non-zero characteristic Lyapunov indicators. In particular, this makes it possible to construct invariant contracting and expanding fibrations passing through almost every point of Ω_k for a frame flow on manifolds satisfying the conditions of Theorems 7.2–7.4 and $\mu(\Lambda) > 0$ (or $\mu(\Lambda_0) > 0$) or the conditions of Theorem 7.5.

Remark 9.2. The method described above for the construction of $\mathfrak{S}^c$ and $\mathfrak{S}^e$ is based on the use of topological properties of $\mathfrak{S}^-$ and $\mathfrak{S}^+$ (contraction of the fibres at a fairly fast rate). Unfortunately, geometric approaches to the construction of $\mathfrak{S}^c$ and $\mathfrak{S}^e$ are unknown: as happens for example in the case of geodesic flows, they would probably permit us to define them in more general cases. The only exceptions are two-dimensional manifolds: a 2-frame flow on a compact surface with a Riemannian metric without conjugate points and with non-positive Euler characteristic has invariant fibrations $\mathfrak{S}^c$ and $\mathfrak{S}'^e$ that can be projected into $\mathfrak{S}^-$ and $\mathfrak{S}^+$ of the geodesic flow, respectively. The fibre of $\mathfrak{S}^c$ (or $\mathfrak{S}^e$) passing through a 2-frame $(v,\ \xi)$ consists of all 2-frames $(w,\ \eta)$, oriented like $(v,\ \xi)$, for which $w \in \mathfrak{S}^-(v)$ (or $w \in \mathfrak{S}^+(v)$).

§10. Topological and ergodic properties of a frame flow

1. The investigation of topological and ergodic properties of frame flows on compact Riemannian manifolds of negative curvature proceeds within the framework of the general theory of partially hyperbolic dynamical systems. At its basis lies the concept of transitivity of the pair of fibrations introduced in [12] (see §4). A pair of continuous fibrations $\mathfrak{S}_1$ and $\mathfrak{S}_2$ on a manifold M is said to be transitive if there exists a natural number N and an $R > 0$ such that for any $x, x' \in M$ there exist $x_1, \ldots, x_N \in M$ for which $x_1 = x$, $x' = x_N$, $x_{i+1} \in \mathfrak{S}_j(x_i)$ ($i = 1, \ldots, N-1$, $j = 1$ or 2) and

$$\rho_{\mathfrak{S}_{j(x_i)}}(x_i, x_{i+1}) < R.$$

It turns out that this property is preserved under sufficiently small perturbations of the partially hyperbolic dynamic system under the condition that the latter has a sufficiently high class of smoothness. Moreover, a partially hyperbolic dynamic system with a transitive pair of fibrations is topologically transitive.

Not every frame flow on a compact Riemannian manifold of negative curvature has a transitive pair of fibrations. A relevant example was constructed by Margulis and described in [12] (see §6). We mention that this flow is not ergodic and has first integrals. However, this situation is untypical.

Theorem **10.1** (see [10], Proposition 3.1). *Let Φ^t be a frame flow on a compact Riemannian manifold M with a metric $\langle,\rangle$ of class C^r, $r \geqslant 2$, of negative curvature. Then $\langle,\rangle$ can be perturbed by a sufficiently small amount within the class C^r of metrics on M such that the pair $\tilde{\mathfrak{S}}^c$ and $\tilde{\mathfrak{S}}^e$ for the frame flow Φ^t of the induced perturbed metric is transitive.*

A simple modification of the proof of this theorem yields the following important result.

Theorem **10.2** (see [10], Theorem 3.1). *In the space of all metrics of negative curvature of class C^r, $r \geqslant 2$, on a manifold M, an open and dense set is formed by those for which the frame flow is topologically transitive for any $k = 1, \ldots, p-1$, where $p = \dim M$.*

2. A frame flow Φ^t on a manifold M has a smooth invariant measure $\varkappa$ induced by the Riemannian metric and given by

$$d\varkappa = d\mu\, d\omega,$$

where μ is the measure in SM (see §2.2) and $d\omega$ is the element of measure on the Stiefel manifold $V_{p-1,\,k-1}$.

Using general results on ergodic properties of partially hyperbolic dynamical systems and Theorem 3.1, we can prove the following assertion.

Theorem 10.3 (see [10], Theorem 3.2). *In the space of all metrics of negative curvature of class C^r, $r \geqslant 3$, on a manifold M, an open and dense set is formed by those for which a k-frame flow is a Y-flow (it can be shown that in this case it is isomorphic to a Bernoulli flow) for any $k = 1, \ldots, p-1$, where $p = \dim M$.*

This last assertion can be strengthened considerably when the dimension of M is odd.

Theorem 10.4 (see [47]). *Let M be a compact Riemannian manifold with a Riemannian metric of class C^r, $r \geqslant 3$, of negative curvature. We assume that $\dim M = p$ is odd and $p \neq 7$. Then a k-frame flow is ergodic and also isomorphic to a Bernoulli flow for any $k = 1, \ldots, p-1$.*(1)

In the case $\dim M = 3$ this result was obtained earlier in [11], and for $\dim M = 5$ it was established by Anosov. In [47] it was proved that when $\dim M = 7$, a 2-frame flow is ergodic.

We mention in conclusion that a k-frame flow on a compact manifold of constant negative curvature has a transitive pair of fibrations, is topologically transitive, ergodic, and isomorphic to a Bernoulli flow (see [12], Proposition 6.5). These same properties hold for a frame flow in a metric sufficiently close (in the class of C^3-metrics) to a metric of constant negative curvature (this was proved in [24] for 2-frame flow).

3. We now consider frame flows on manifolds without focal points. If the conditions of Theorem 7.5 are satisfied, then a geodesic flow has almost everywhere non-zero characteristic Lyapunov indicators, so that the corresponding frame flows can be presented as G-extensions over systems with non-zero Lyapunov indicators (see Remark 9.1). By using this circumstance we can prove the following assertion by the methods developed in [46].

Theorem 10.5. *If the conditions of Theorems 7.2–7.4 are satisfied and $\mu(\Lambda) > 0$, or if the conditions of Theorem 7.5 are satisfied, then a k-frame flow on a manifold M has a finite number of ergodic components $A_1, \ldots, A_l$ such that $\pi(A_i) = SM$ ($i = 1, \ldots, l$) and there exist subgroups $G_1, \ldots, G_l$ of $SO(k-1)$ such that for almost every $v \in SM$ the subgroup G_i corresponds to the intersection $\pi^{-1}(v) \cap A_i$. Moreover, the restriction $\Phi^t | A_i$ is isomorphic to a Bernoulli flow.*

(1) Anosov pointed out to me the following fact: as is clear from [11] the question of the ergodicity of a k-frame flow reduces to that of the construction of the structure group of the bundle of $(p-2)$-frames over a $(p-1)$-dimensional sphere; the latter is treated fully in [86], and this allows us to obtain the conclusion of Theorem 10.4 (at least in that part that refers to ergodicity for $p \neq 7$).

§11. Definition of the horocycle flow

1. The first significant results in the study of horocycle flows were obtained by Hopf and Hedlund in the 30's (see [44] and [72]). They examined compact connected orientable surfaces of constant negative curvature.(1) Their research was continued in [15], [31], and [63] (see also the recent papers [48], [97]-[99], [105]). The approach developed by these authors was algebraic and based on the presentation of horocycle flows as actions of a one-parameter group on a compact homogeneous space, which allowed them to use the representation theory of Lie groups etc.(2)

However, such a good algebraic structure is intrinsic in horocycle flows only in a metric of constant negative curvature, and attempts to transfer the results to more general classes of Riemannian metrics (for example, to those having variable negative curvature) led to the development of new methods based on a wide use of the hyperbolic properties of geodesic flows. The relevant results were obtained by Marcus, Bowen, Eberlein, and, in part, Margulis (see [89]-[92], [49], [55], [27]).

2. **The construction of the horocycle flow.**
Let M be a compact orientable connected surface with non-positive Euler characteristic and a Riemannian metric without conjugate points; let H be the universal Riemannian covering manifold.

For each $x \in H$ and $p \in H(\infty)$ the limiting sphere $L(x, p)$ is a smooth curve in H, which can be parametrized by the "arc length" (the "length" is induced by the Riemannian metric). We introduce a "direction" on the limiting sphere $L(x, p)$ by specifying for $y \in L(x, p)$ a unit vector $V_{L(x,p)}(y)$ touching $L(x, y)$ at y and directed so that the pair $\{V_{L(x,p)}(y), v(y)\}$ (where $v(y) \in SM$ is the vector for which $\pi(v(y)) = y$ and $\gamma_{v(y)}(+\infty) = p$) is positively orientated. The fact that this "direction" can be prescribed continuously by the method given above follows from the properties of limiting spheres (see §5.3 and also [55], Proposition 2.1). Thus, on each $L(x, p)$, we have a concept of "left and right". We now define a map h: $SM \times \mathbf{R} \to SH$ by putting

$$h(v, 0) = v, \text{ when } t = 0$$

$$h(v, t) = w, \text{ when } t \neq 0$$

(1) As for the existence of horocycles and the minimality of the horocycle bundle, they were known also for the case of variable negative curvature.

(2) In this presentation: 1) $M = \Gamma/G$, where $G = SL(2, \mathbf{R})$, $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and Γ is a discrete subgroup of G; 2) the Riemannian metric is left-invariant in G; 3) the horocycle flow h^t on M is defined by

$$h^t(\Gamma g) = \Gamma g \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad g \in G, \quad t \in \mathbf{R}.$$

For details, see [22].

where $w \in SM$ is the vector such that

1) $\gamma_v(+\infty) = \gamma_w(+\infty) = p$,
2) $y = \pi(w) \in L(x, p)$, where $x = \pi(v)$,
3) $\rho_{L(x, p)}(x, y) = t$,
4) y lies to the right of x.

We now define the horocycle flow h^t: $SH \to SH$ by putting

$$h^t(v) = h(v, t).$$

It follows directly from these definitions that for any s and t

$$h^{s+t} = h^s \circ h^t. \tag{11.1}$$

Since the map h is continuous, this means that h^t is a continuous flow in SH. It can be projected into the horocycle flow in SM (also denoted by h^t). Since its trajectories are smooth curves in SM, the vector field $Z(v) = \frac{d}{dt} h^t(v) \mid_{t=0}$ is well-defined.

Proposition **11.1** (see [55], §2, or [76]). 1) *The vector field Z is continuous, and for any $v \in SM$ there is a unique curve in Z passing through v.*

2) *If $K_x(P) < 0$ for all x and P, then*(1) $Z \in C^1$, *and* $\langle Z(v), V(v)\rangle = 0$ *for all $v \in SM$ (where $V(v)$ is the vector field giving the geodesic flow, and $\langle , \rangle$ is the canonical Riemannian metric in TTM (see* §1.2).

Between geodesic and horocycle flows there exists the following important relation: for any $v \in SM$ and $s, t \in \mathbf{R}$

$$(f^t \circ h^s)(v) = (h^{s^*} \circ f^t)(v), \tag{11.2}$$

where $s^*(t, s, v)$: $\mathbf{R} \times \mathbf{R} \times SM \to \mathbf{R}$ is a certain function connected with the given horocycle flow. For manifolds of zero curvature ($K_x(P) \equiv 0$) we have $s^*(t, s, v) = s$ and for manifolds of constant negative curvature $s^*(t, s, v) = se^{-\sqrt{-K}t}$.

We have defined the horocycle flow using $\mathfrak{S}^-$; the horocycle flow whose trajectories are the fibres of $\mathfrak{S}^+$ can be constructed analogously. The function s^* in (11.2) is "expanding": in the case of constant negative curvature, $s^*(t, s, v) = se^{\sqrt{-K}t}$.

3. Generalized horocycle flows.

The construction above of the horocycle flow has the following far-reaching generalization. Let f^t be a smooth flow on a compact Riemannian manifold M and W a stable invariant orientable one-dimensional C^1-fibration. Each fibre $W(x)$ can be parametrized by the "arc length" and the concept of "left and right" is well-defined and gives a direction field on $W(x)$. Let β: $M \times \mathbf{R} \to M$ be a *W-parametrization*, that is, a constant map such that for any $x \in M$ the function $\beta(s, t) \in W(x)$ is strictly monotonic. We define a continuous flow h^t by putting $h^t(x) = y$, where y is a point on $W(x)$ to the right of x, such

(1)Here it is essential that we consider only compact manifolds; for non-compact manifolds this assertion, generally speaking, is not true.

that $\rho_{W(x)}(x, y) = \beta(x, t)$. It is easy to check that (11.1) holds for any s and t and the trajectories of h^t are the fibres of W. Then h^t is called a generalized horocycle flow (sometimes we speak loosely of a *horocycle flow*) or *W-flow*. It is uniquely connected with the W-parametrization of M. The flows h^t and f^t are connected by (11.2), where the function s^*: $\mathbf{R} \times \mathbf{R} \times \times M \to \mathbf{R}$ is determined by the given W-parametrization. We mention two special W-parametrizations:

1) *uniformly expanding*, for which

$$s^*(t, s, x) = \lambda^t s \tag{11.3}$$

for some $\lambda > 1$;

2) *standard*, for which

$$\beta(x, t) = y,$$

where y is a point in $W(x)$ to the right of x, for which $\rho_{W(x)}(x, y) = t$, that is, each fibre $W(x)$ is parametrized by the "arc length".

***Theorem* 11.1** (see [91], Corollary 6.3). *Let f^t be a Y-flow on a compact Riemannian manifold M, and* $\dim W^+(x) = 1$ (*or* $\dim W^-(x) = 1$). *Then there eixsts a uniformly expanding W^+ (or W^-) parametrization for which* $\log \lambda = h(f)$ (*the topological entropy of f^t*). *If f^t is topologically mixing* (*so that the W^+- and the W^--flows are minimal*), *then the parametrization is uniquely determined* (*to within a scalar change of time*).

A uniformly expanding W^+-parametrization is continuous (see [89]). Flows for which it is smooth form a nowhere dense set in the space of all Y-flows of class C^2. (This and various sharper assertions are established in [91], §6.)

In what follows we shall understand by a W^+-flow a generalized horocycle flow corresponding to a certain parametrization of the one-dimensional extending invariant fibration of a Y-flow.

When f^t is a geodesic flow, then we take for W the fibration $\mathfrak{S}^+$ (we could equally well take $\mathfrak{S}^-$; for brevity we call the corresponding generalized horocycle flows $\mathfrak{S}^+$-and $\mathfrak{S}^-$-flows). The horocycle flow introduced above corresponds to the standard parametrization. For a metric of constant negative curvature the standard and the uniformly expanding parametrization are the same.

§12. Topological and ergodic properties of the horocycle flow

1. Minimality.

Let M be a connected orientable surface with non-positive Euler characteristic and a Riemannian metric without conjugate points; and let f^t be a geodesic flow in SM and h^t a horocycle flow (with any parametrization). We denote by Ω_k the set of non-wandering points of h^t.

Theorem 12.1. *If M is compact, then h^t is minimal.*

This proposition for Riemannian metrics of negative curvature was known to Hopf and Hedlund (see [44] and [72]), and was proved in full generality by Eberlein in [55].

For compact surfaces of negative curvature Theorem 12.1 follows from a more general assertion (see [3], §4).

Theorem 12.2. *If f^t is a topologically mixing Y-flow on a compact p-dimensional Riemannian manifold and* $\dim W^+(x) = 1$ (*or* $\dim W^-(x) = 1$), *then any W^+- (or W^--) flow is minimal.*

From Theorem 12.1 it follows that any trajectory of the horocycle flow h^t is dense in SM, and the flow has no periodic trajectories.

We say a few words on non-compact manifolds. In [55] Eberlein proved an assertion stronger than Theorem 12.1.

Theorem 12.3 (see [55], Theorem 4.5 and Proposition 4.8). *If the horocycle flow on a surface M is minimal, then this surface is compact. Moreover, if $A \subset \Omega_h$ is a non-empty compact and minimal set, then either $A = SM$ (in particular, M is compact) or M is non-compact and A coincides with a certain periodic trajectory.*

In particular, if M is non-compact, then the horocycle flow has periodic trajectories. Moreover, if, in addition, M is finitely-connected, then any minimal set in Ω_h coincides with some periodic trajectory (see [55], Corollary 4.7).

2. Topological transitivity and mixing.

Theorem 12.4 (see [55], Theorem 4.1 and Corollary 4.2). 1) *If $\Omega_h = SM$ (in particular, if M is compact), then a horocycle flow h^t is topologically transitive. If, in addition, M is finitely connected, then any trajectory $\{h^t(x)\}$ is either dense in M or periodic.*

2) *If there exist $v_1, v_2, \in SH$ such that the trajectories $\{f^t(v_1)\}$ and $\{f^t(v_2)\}$ are periodic with periods T_1 and T_2, respectively, and the ratio T_1/T_2 is irrational, then $\Omega_h = SM$ and the flow $h^t | \Omega_h$ is topologically transitive.*

It is quite obvious that such properties as minimality and transitivity do not depend on the choice of the $\mathfrak{S}^+$- or $\mathfrak{S}^-$-parametrization. The matter is different with the property of topological mixing, the presence of which depends a priori on the choice of parametrization. Nevertheless the following theorem holds.

Theorem 12.5 (see [91], Theorem (3.2)). *Any minimal W^+-flow is topologically mixing. In particular, any $\mathfrak{S}^+$- or $\mathfrak{S}^-$-flow on a compact connected orientable surface of negative curvature is topologically mixing.*

This was generalized by Eberlein to a Riemannian metric of non-positive curvature.

Theorem 12.6 (see [55], Theorem 4.11). *Let M be a compact connected orientable surface with negative Euler characteristic and non-positive curvature. Then a horocycle flow (in the standard parametrization) is topologically mixing.*

3. The invariant measure for the horocycle flow.

Theorem 12.7. *Any W^+-flow f^t has an invariant Borel measure (denoted by μ_h). This measure is unique if h^t is minimal.*

This was proved(1) by Marcus in [90], but it follows easily in essence from results of Margulis (see [27] and [28], and also Plante [96]). In [49] Bowen and Marcus generalized it to W^+-flows connected with invariant fibrations of a flow satisfying axiom A.

We now give a brief description of the measure μ_h (see [91], §2). Suppose that $x \in M$, and that a, b, and ε be sufficiently small positive numbers. Also, let $U_\varepsilon^+(x)$ be the ball on $W^+(x)$ with centre at x and radius ε. We put

$$V = \bigcup_{r\in[0,\, a]} h^r \Big(\bigcup_{t\in[0,\, b]} f^t (U_\varepsilon^+ (x))\Big).$$

Then V is a neighbourhood of x in M and there exists a natural homeomorphism χ: $[0, a] \times [0, b] \times U_\varepsilon^+(x) \to V$ given by

$$\chi(r, t, y) = h^r \circ f^t(y).$$

Proposition 12.1 (see [91], Proposition 2 or [90], §4). *Let g: $V \to \mathbf{R}$ be a continuous function. Then, when we multiply the identities by χ, we obtain*

$$\int_V g(y)\, d\mu_h(y) = \int_{U_\varepsilon^+(x)} \int_0^b \int_0^a \lambda^{-t} (h^r \circ f^t (z))\, dr\, dt\, d\mu^+ (z),$$

where $\lambda = e^h > 1$ is the constant in (11.3) (the existence of λ is established in Theorem 11.1) and μ^+ is a finite Borel measure(2) on $U_\varepsilon^+(x)$.

The following assertion establishes one remarkable property of a uniformly expanding W^+-parametrization.

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(1)Analogues of this assertion for W^+-flows connected by invariant foliations of Y-diffeomorphisms and diffeomorphisms satisfying axiom A can be found in [41] and [100].

(2)Indeed, there exists a compatible uniformly expanding system of Borel measures $\{\mu_x^+\}$ on $\{W^+(x)\}$, where $\mu^+ = \mu_x^+ | U_\varepsilon^+(x)$ (uniform expansion means that $\mu^+_{f^t(x)}(f^t(A)) = \lambda^t \mu_x^+(A)$, $A \subset W^+(x)$ is a Borel set, and compatibility that $|\mu_y^+(A) - \mu_z^+(B)| \leqslant \delta\mu_y^+(A)$ for any two δ-canonically isomorphic Borel sets $A \subset W^+(y)$ and $B \subset W^+(z)$; see [91], §6). The existence of such a system of measures can be proved by various methods [27], [28], [90], [49]. The measures μ_x^+ are non-atomic, σ-finite, and positive on open sets (these facts lie at the basis of the proof of Theorem 11.1).

Theorem 12.8 (see [91], Remark (6.4)). *If a W^+-flow is minimal, then an invariant measure for a uniformly expanding W^+-parametrization is also invariant for the flow f^t and coincides with the measure of maximum entropy for f^t (denoted by μ_f^0).*

The converse is also true.

Theorem 12.9 (see [91], Proposition (6.8)). *Suppose that*: 1) *h^t is a minimal W^+-flow*; 2) *the function $s^*(t, s, x)$ is continuously differentiable*(1) *in s*; 3) *the flows h^t and f^t preserve a common finite measure. Then the W^+-parametrization corresponding to h^t is uniformly expanding.*

From the assertions above it follows that the horocycle flow on a compact connected orientable surface of negative curvature has a unique invariant measure. When the curvature is constant (and negative), this measure is smooth and invariant for a geodesic flow with maximum entropy.(2)

4. Ergodic properties.

It follows directly from Theorem 12.7 and criteria of strict ergodicity(3) that

Theorem **12.10** (see [90], [91]). *Any minimal W^+-flow is ergodic. In particular, if g is a continuous function on M, then*

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t g(h^s(x))\,ds=\int h(x)\,d\mu_h(x)$$

uniformly in $x\in M$.

In [91] Marcus established the mixing property for W^+-flows.

Theorem 12.11 (see [91], §4). *A minimal W^+-flow h^t is mixing if one of the following conditions is satisfied*:

1) *$\partial^2 s^*/\partial t\partial s$ exists and is continuous in t, s, and z*;
2) *$f^t\in C^2$, and the W^+-parametrization determining h^t is standard*;
3) *the W^+-parametrization determining h^t is uniformly expanding.*

Each of the last two conditions implies the first.

We now consider the question of the topological entropy of a W^+-flow. In [91] Marcus proved, in effect, the following assertion.

Theorem 12.12 (see [91], proof of Theorem (5.1)). *Any minimal smooth W^+-flow has zero topological entropy (and consequently zero metric entropy in the measure μ_h).*

---

(1)This condition is automatically satisfied if $f^t\in C^2$ (see [91], Corollary (4.3).

(2)Green proved in [69] that if the curvature is not constant, then the invariant measure for the horocycle flow is not smooth.

(3)By strict ergodicity of a flow we mean the existence of a unique invariant measure (which then is ergodic). Sometimes, in addition, minimality of the flow is assumed (which in our case is also true).

These results have the following consequence.

***Theorem* 12.13.** *The horocycle flow on a compact connected orientable surface of negative curvature is strictly ergodic, mixing, and has zero topological entropy.*

Recently Gura proved that this flow has the property of separating the trajectories (unpublished).

For the case of metrics of constant negative curvature the strict ergodicity of a horocycle flow was established earlier by Furstenberg in [63], and the fact that the entropy is zero by Gurevich in [18]. Moreover, Parasyuk proved in [31] that this flow had a Lebesgue spectrum. Ratner proved in [99] that a horocycle flow is standard (this is true for any W^+-flows). It is interesting to mention that its Cartesian square is no longer standard (see [98]) and this flow gives so far the only "natural" (and smooth) example of this kind.

Finally, in [92] Marcus proved that the horocycle flow (in the standard parametrization) mixes to any degree of multiplicity in a metric of constant negative curvature (this is also true for Riemannian metrics of variable negative curvature provided that the parametrization is uniformly expanding).

We give one further result of Ratner for the horocycle flow on a connected orientable surface of negative curvature.

***Theorem* 12.14** (see [97]). *Let h^t, and h_2^t be two horocycle flows on manifolds $M_1 = \Gamma_1/G$ and $M_2 = \Gamma_2/G$ (see §11.1, footnote(1)), where the subgroups $\Gamma_1, \Gamma_2 \subset G$ are hyperbolic (that is, $|\mathrm{tr}\, g| > 2$ for any $g \in \Gamma_1$ or Γ_2, $g \neq e, -e$). Then the flows h_1^t and h_2^t are isomorphic (that is, there exists a metric isomorphism Ψ: $M_1 \to M_2$ such that $\Psi h_1^t(x) = h_2^t(\Psi(x))$ for almost all $x \in M$, and $t \in \mathbf{R}$) if and only if $\Gamma_2 = g\Gamma_1 g^{-1}$ for some $g \in G$.*

5. Horosphere flows.

In the case of metrics of constant negative curvature an analogue of a horocycle flow can be defined for manifolds of dimension greater than 2. The corresponding flow (denoted by Φ^t) acts in the space of 2-frames on the manifold M: $\Phi^t(v, w) = (v', w')$, where v' and w' are obtained by a parallel shift of the vectors v and w along the geodesic in $L(v)$ defined by the vector w (touching $L(v)$ at $\pi(v)$) at the time t. This definition is due to Hopf (see [44]). Another definition, using an algebraic approach is a generalization of that in §11.1, footnote(1) and consists in the following: horosphere flows are actions of certain one-parameter subgroups on the homogeneous space of a locally compact Lie group (see [24], Ch. 4, §4). The ergodic properties of such flows have been studied by Veech [105], who proved that they are strictly ergodic; certain generalizations of his results were obtained by Bowen [48].

An attempt (as yet incomplete) has been made to introduce the concept of a horocycle flow for multi-dimensional manifolds of variable negative curvature: a vector field is constructed, but its smoothness has not been proved (see [68]).

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